10. Log resolutions

Definition 10.1. *Let X be normal variety and let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf on X. We say that $\mathcal{I}$ is* ***principal*** *if X is smooth and every point of X has a neighbourhood with coordinates $x_1, x_2, \dots, x_n$ so that $\mathcal{I}$ is locally given by a single monomial.*

We have the following celebrated result of Hironaka:

Theorem 10.2 (Principalisation of Ideals)**.** *Let M be a smooth variety and let $\mathcal{I}$ be an ideal sheaf on X.*

Then there is a composition of smooth blow ups $\pi\colon Y \longrightarrow X$ along smooth centres, with support contained in the support of $\mathcal{O}_X/\mathcal{I}$, such that $\pi^*\mathcal{I}$ is a principal ideal.

Definition 10.3. *Let (X, Δ) be a log pair.*

We say that (X, Δ) is ***log smooth****, if the pair (X, D) has global normal crossings (that is every irreducible component of D is smooth and locally (in the analytic or étale topology) about any point of X, $(X, D = \sum \Delta_i)$ is isomorphic to $(\mathbb{C}^n, H_1 + H_2 + \cdots + H_k)$ where $H_1, H_2, \dots, H_n$ are the coordinate hyperplanes).*

A ***log resolution*** *of (X, Δ) is a birational morphism $\pi\colon Y \longrightarrow X$ such that $(Y, \Gamma = f_*^{-1}\Delta + E)$ is log smooth, where $f_*^{-1}\Delta$ is the strict transform of Δ and E is the sum of the exceptional divisors, and there is a divisor F, supported on the exceptional locus, such that F is π-ample.*

Remark 10.4. *Note that in the definition of a log resolution, we make no requirement that the locus where π is not an isomorphism is concentrated over any special locus in X (such as where X is singular).*

Corollary 10.5. *Every log pair (X, Δ) has a log resolution.*

Proof. Embed $X \subset M$ inside a smooth variety, where X has codimension at least two. Let $\pi\colon N \longrightarrow M$ be a birational morphism which principalises $\mathcal{I}_X \subset \mathcal{O}_M$. Then the inverse image of X is a divisor. Then at some stage X must have been contained in a centre of some blow up. But the first such time this happens, the centre must be X itself.

In particular, we can resolve the singularities of X. So replacing X by its resolution, and Δ by the strict transform of Δ plus the exceptional locus, we may assume that X is log smooth. Now apply (10.2) to $\mathcal{I}_D \subset \mathcal{O}_X$. □

Theorem 10.6 (Elimination of indeterminancy)**.** *Let $\phi\colon X \dashrightarrow Y$ be a rational map between projective varieties.*

Then there are morphisms $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$, where p is a composition of smooth blow ups along smooth centres, W is smooth and there is a commutative diagram

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \overset{\phi}{\dashrightarrow} & Y. \end{array}$$

Moreover if X is smooth and $y \in Y$ is in the image of the indeterminancy locus of ϕ then there is a non-constant morphism $f\colon \mathbb{P}^1 \longrightarrow Y$ such that $f(0) = y$.

Proof. Pick an embedding of $Y \subset \mathbb{P}^n$ into projective space and let H be a hyperplane section. Let $\phi^*H = M + F$ be the decomposition of ϕ^*H into its fixed and mobile parts. Then the linear system $|M|$ defines ϕ. Let B the scheme theoretic base locus of $|M|$. Then $B = 0$ if and only if ϕ is a morphism (or perhaps better, extends to a morphism). Note that the codimension of B is at least two.

Let $\mathcal{I}_B$ be the ideal sheaf of B. Let $p\colon X \longrightarrow Y$ be the birational morphism, whose existence is guaranteed by (10.2). Let $q\colon W \dashrightarrow Y$ be the induced rational map. Then $q^*H = M_1 + F_1$ is the decomposition of q^*H into its mobile and fixed parts, where $F_1 = p^*B$ and $M_1 = p^*M - F_1$. But then $|M_1|$ is base point free, so that q is a morphism.

Let $V \subset X$ be the indeterminancy locus of ϕ, and let $Z = qp^{-1}(V)$. If $x \in V$, them the $qp^{-1}(x)$ is positive dimensional. Since the image of a rationally chain connected variety is rationally chain connected it suffices to prove that the fibres of p are rationally chain connected. We prove this by induction on the number of blow ups. Suppose that p factors as $p_1\colon W_1 \longrightarrow X$ and $\pi\colon W \longrightarrow W_1$, where π is a smooth blow up of $B \subset W_1$. By induction the fibres of p_1 are rationally chain connected. Let $E_1 \subset W$ be the intersection of the exceptional divisor E with a fibre of p and let $B_1 \subset W_1$ be the image of E_1. Then the fibres of E_1 over B_1 are projective spaces, which are rationally connected. If f_1 and f_2 are two points of two fibres F_1 and F_2 then pick x_1 and x_2 belonging to the fibres and the strict transform G of B_1. Then we can find a rational curve connecting f_1 to x_1 in F_1, a chain of rational curves connecting x_1 to x_2 in G and a rational curve connecting x_2 to f_2 in F_2. The resulting chain connects f_1 to f_2 in the fibre E_1. □

To get some idea of the proof of (10.6), consider the case of smooth projective surfaces. To emphasize this point, we change notation and consider $\phi\colon S \dashrightarrow Y$, where S is a smooth projective surface. As M is mobile it is nef (here is one important place where we use the fact that

S is a surface). We proceed by induction on $d = M^2 \geq 0$. Suppose that ϕ is not defined at $x \in |M|$. Then x is a base point of $|M|$. Let $\pi\colon S_1 \longrightarrow S$ be the blow up. Let $\phi_1\colon S_1 \dashrightarrow Y$ be the induced rational map. If ϕ_1 is given by M_1, then

$$M_1 = \pi^* M - mE,$$

where $m > 0$ is a positive integer (in fact $\pi^* M = M_1 + mE$ gives the decomposition into fixed and mobile parts). Now

$$M_1^2 = (\pi^* M - mE)^2 = d - m^2 < d.$$

Thus we are done by induction on d.

Let $X = \mathbb{C}^3$ and let X_1 be the blow up of the origin of X. The exceptional divisor is then a copy of $\mathbb{P}^2$. Let X_2 be the blow up of X_1 along an smooth cubic in the exceptional divisor. Then the exceptional locus is a copy E of $\mathbb{P}^2$ joined to a $\mathbb{P}^1$-bundle F over an elliptic curve, joined along a section and a cubic. Then $E \cup F$ is a rationally chain connected variety, and yet F is not rationally connected. To connect two points of F, f_1 and f_2, let F_1 and F_2 be the two fibres which contain them. Now let x_1 and x_2 be the points in E which meets these two fibres. Let l be the line connecting x_1 to x_2. Then $F_1 \cup l \cup F_2$ connects f_1 to f_2.

Example 10.7. *Let*

$$X = C \times \mathbb{P}^2 \cup \mathbb{P}^2 \times C \subset \mathbb{P}^2 \times \mathbb{P}^2.$$

Then X is rationally chain connected, but neither component is rationally connected.