THIRD HOMEWORK, DUE WEDNESDAY APRIL 29TH

1. Show that if C is a smooth curve and p, q and r are smooth points of C (possibly equal) then $K_C + p + q$ is free, $K_C + p + q + r$ is very ample but that $K_C + p + q$ is never very ample.
2. Let X be a projective variety and let H and D be two $\mathbb{Q}$-Cartier divisors.
(i) If D is base point free and H is very ample divisor then show that $D + H$ is very ample.
(ii) If D is semiample and H is ample then show that $D + H$ is ample.
(iii) Using (ii), give an alternative proof of the fact that if H is ample then there is integer m_0 such that $D + mH$ is ample for all $m \geq m_0$.
3. Let D be a divisor. Show that there are divisors $F \geq 0$ and M such that $D = M + F$ and

$$|D| = |M| + F.$$

and the base locus of M has codimension two or more. F is called the **fixed divisor** and M is called the **mobile part** of D. If M is free, show that $\phi_{|D|}\colon X \dashrightarrow \mathbb{P}^N$ extends to a morphism $\phi_{|M|}\colon X \longrightarrow \mathbb{P}^N$. In particular give another proof that every rational $\phi\colon C \longrightarrow Y$ from a smooth curve to a projective variety Y always extends to a morphism.
4. Let Y be a hypersurface of degree d in $\mathbb{P}^{n+1}$ and let l be a line intersecting X in d points $p_1, p_2, \dots, p_d$. Let $\pi\colon Y \longrightarrow X$ be the blow up of X along the first $d-1$ points $p_1, p_2, \dots, p_{d-1}$ with exceptional divisors $E_1, E_2, \dots, E_{d-1}$. Let L be the line bundle

$$L = \mathcal{O}_X(\pi^* H - \sum_{i=1}^{d-1} E_i) = \mathcal{O}_X(D),$$

where H is a hyperplane in $\mathbb{P}^n$.
(i) Show that L is nef and big.
(ii) Show that L is not ample if there is a line m contaned in X passing through p_1.
(iii) Show that the base locus of $|mL|$ is equal to p_d.
(iv) Show that

$$\mathcal{O}_X(K_X + nD) = \mathcal{O}_X((d-2)\pi^* H - \sum_{i=1}^{d-1} E_i)$$

has a base point at p_d. (In fact L is ample if X is general and d is sufficiently large).

Challenge problems:

5. Finish the proof of asymptotic Riemann Roch given in class (that is, identify the second term).