PRACTICE PROBLEMS FOR THE FINAL

Here are some practice problems for the final. They contain bookwork plus some problems culled from various locations.

1. Let γ be a closed curve. Given $a \in \mathbb{C} - \gamma$, define the winding number $n(\gamma; a)$. Show that the winding number takes on only finitely many values.

2. Let U be a region. Show that f is holomorphic on U if and only if it is analytic on U.

3. Let f and g be two holomorphic functions on a region U. Let

$$E = \{\, a \in U \mid f(a) = g(a) \,\}.$$

Show that $E = U$ if and only if there is a point $a \in U$ which is an accumulation point of E.

4. Let f be a holomorphic function on a region U. If $a \in U$ then show that there is an integer n and a holomorphic function $g(z)$ on U such that $f(z) = (z-a)^n g(z)$, where $g(a) \neq 0$.

5. Let f be an entire function. Show that f has a pole at infinity if and only if f is a polynomial.

6. Evaluate the following integrals:

(i)

$$\int_0^{2\pi} \frac{\cos(3\theta)\,\mathrm{d}\theta.}{5 - 4\cos(\theta)}$$

(ii)

$$\int_0^{\infty} \frac{\mathrm{d}x.}{x^6 + 1}.$$

7. State and prove the open mapping theorem.

8. Find all complex numbers such that $\sec z = i$.

9. Find the residue of

$$\frac{e^{z^2}}{z^n},$$

at all of its poles, for all values of $n > 0$.

10. (i) Compute the residue at the origin of

$$\frac{e^z}{\sin^2 z}.$$

(ii) Compute

$$\int_\gamma e^{1/z}\,\mathrm{d}z,$$

where γ is a closed curve that does not contain the origin.