PRACTICE PROBLEMS FOR THE MIDTERM

Here are some practice problems for the first midterm culled from various locations (several problems are a bit more involved than the midterm problems but are hopefully useful for review):

1. Let $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function. If $z_0 \in U$ is a point such that $f'(z_0) \neq 0$ then show that f preserves angles between smooth curves intersecting at z_0.

Find a biholomorphic map between the two regions U and V, where U is the second quadrant of the unit disc,

$$U = \{\, z \in \mathbb{C} \,|\, |z| < 1, \pi/2 < \arg(z) < \pi \,\}$$

and V is the area outside the unit disc of the first quadrant:

$$V = \{\, z \in \mathbb{C} \,|\, |z| > 1, 0 < \arg(z) < \pi/2 \,\}.$$

2. Let $f(z)$ be an entire function. State Cauchy's integral formula, relating the nth derivative of f at a point a with the values of f on some circle around a.

State Liouville's theorem, and deduce it from Cauchy's integral formula.

Suppose that for some k we have that $|f(z)| \leq |z|^k$ for all z. Prove that f is a polynomial.

3. What is the radius of convergence of the power series

$$\sum_{n=1} \frac{z^n}{n}?$$

4. Find a conformal transformation $f(z)$ that maps the region

$$U = \{\, z \in \mathbb{C} \,|\, 0 < \arg(z) < \frac{3\pi}{2} \,\}$$

onto the strip

$$V = \{\, z \in \mathbb{C} \,|\, 0 < \mathrm{Im}(z) < 1 \,\}.$$

Hence find a bounded harmonic function ϕ on U subject to the boundary conditions $\phi = 0$ on $\arg z = 0$ and $\phi = A$ on $\arg z = 3\pi/2$ for some real constant A.

5. Using Cauchy's integral formula, write down the value of a holomorphic function $f(z)$ where $|z| < 1$ in terms of a contour integral around the unit circle, $\zeta = e^{i\theta}$.

By considering the point $1/\bar{z}$ show that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{1-|z|^2}{|\zeta - z|^2}\,\mathrm{d}\theta.$$

By setting $z = re^{i\alpha}$, show that for any harmonic function $u(r,\alpha)$,

$$u(r,\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)\frac{1-r^2}{1-2r\cos(\alpha-\theta)+r^2}\,\mathrm{d}\theta.$$

Assuming that the harmonic conjugate $v(r,\theta)$ can be written as

$$v(r,\alpha) = v(0) + \frac{1}{\pi}\int_0^{2\pi} u(1,\theta)\frac{r\sin(\alpha-\theta)}{1-2r\cos(\alpha-\theta)+r^2}\,\mathrm{d}\theta,$$

deduce that

$$f(z) = iv(0) + \frac{1}{2\pi}\int_0^{2\pi} u(1,\theta)\frac{\zeta+z}{\zeta-z}\,\mathrm{d}\theta.$$

6. Let U be the disc centred at a with radius r and let $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function. Using Cauchy's integral formula, show that for every $0 < s < r$,

$$f(a) = \int_0^1 f(a + se^{2\pi i t})\,\mathrm{d}t.$$

Deduce that if

$$|f(z)| \leq |f(a)| \qquad \text{for every} \qquad z \in U,$$

then f is constant.

Now specialise to the case when $a = 0$ and $r = 1$, so that U is the unit disc. If $f(0) = 0$ and

$$f\colon U \longrightarrow U$$

then show that

$$|f(z)| \leq |z| \qquad \text{for every} \qquad z \in U.$$

Moreover if $|f(w)| = |w|$ for some $w \neq 0$ then there exists λ with $|\lambda| = 1$ such that

$$f(z) = \lambda z \qquad \text{for every} \qquad z \in U.$$