MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. If we write out the Taylor series of $e^z - 1$ we see that $e^z - 1$ has a zero of order one at the origin. But then

$$\frac{e^z - 1}{z}$$

is a holomorphic function which takes the value one at the origin. Thus

$$\frac{z}{e^z - 1}$$

is a holomorphic function which takes the value one at the origin. But then

$$\frac{1}{e^z - 1}$$

has a simple pole at the origin and the residue is one.
Hence

$$g(z) = \frac{1}{e^z - 1} - \frac{1}{z} = \frac{1 - z - e^z}{z(e^z - 1)}$$

is an entire function.
We calculate the first few terms of its Taylor series. The numerator has an expansion

$$-\frac{z^2}{2} - \frac{z^3}{3!} - \dots$$

and the denominator has an expression

$$z^2 + \frac{z^3}{2} + \dots.$$

Taking the ratio we see that

$$g(0) = -\frac{1}{2}.$$

It follows that we can write

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + h(z),$$

where $h(z)$ is entire.
Note that

$$\frac{1}{e^z - 1} + \frac{1}{2} = \frac{1 + e^z}{2(e^z - 1)},$$

is an odd function. Thus $h(z)$ is an odd function. So the Taylor series of $h(z)$ does have the form

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^{2k-1}=\frac{B_1 z}{2}-\frac{B_2 z^3}{4!}+\frac{B_3 z^5}{6!}+\dots.$$

If we multiply through by e^z-1 we have

$$1=(z+\frac{z^2}{2}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}+\frac{z^7}{7!}+\dots)\left(\frac{1}{z}-\frac{1}{2}+\frac{B_1 z}{2}-\frac{B_2 z^3}{4!}+\frac{B_3 z^5}{6!}+\dots\right).$$

Looking at the quadratic terms we get:

$$\frac{B_1}{2}-\frac{1}{4}+\frac{1}{6}=0 \qquad \text{so that} \qquad B_1=\frac{1}{6}.$$

Looking at the quartic terms we get:

$$-\frac{B_2}{4!}+\frac{1}{6\cdot 6\cdot 2}-\frac{1}{2}\frac{1}{4!}+\frac{1}{5!}=0.$$

Thus

$$B_2=\frac{1}{5}+\frac{1}{3}-\frac{1}{2}=\frac{1}{30}.$$

Looking at the sextic terms we get:

$$\frac{B_3}{6!}-\frac{1}{30\cdot 4!\cdot 3!}+\frac{1}{6\cdot 2\cdot 5!}-\frac{1}{2\cdot 6!}+\frac{1}{7!}=0.$$

Thus

$$B_3=\frac{1}{6}-\frac{1}{7}=\frac{1}{42}.$$

2. We have

$$\begin{aligned}\cot z&=\frac{\cos z}{\sin z}\\&=i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}\\&=i\frac{e^{2iz}+1}{e^{2iz}-1}\\&=i+\frac{2i}{e^{2iz}-1}\\&=i+2i\left(\frac{1}{2iz}-\frac{1}{2}+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}(2iz)^{2k-1}\right)\\&=\frac{1}{z}-2\sum_{k=1}^{\infty}\frac{B_k}{(2k)!}(2z)^{2k-1}.\end{aligned}$$

For $\tan z$ we have

$$\begin{aligned}
\tan z &= \frac{\sin z}{\cos z} \\
&= \frac{\sin^2 z}{\sin z \cos z} \\
&= \frac{2\sin^2 z}{\sin 2z} \\
&= -i\frac{e^{2iz} - 2 + e^{-2iz}}{e^{2iz} - e^{-2iz}} \\
&= -i - 2i\frac{e^{-2iz} - 1}{e^{2iz} - e^{-2iz}} \\
&= -i - 2i\frac{1 - e^{2iz}}{e^{4iz} - 1} \\
&= -i + 2i\left(\sum_{k=1}^{\infty} \frac{(2iz)^k}{k!}\right)\left(\frac{1}{4iz} - \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}(4iz)^{2k-1}\right) \\
&= -i + \left(\sum_{k=1}^{\infty} \frac{(2iz)^k}{k!}\right)\left(\frac{1}{2z} - i - \sum_{l=1}^{\infty} 2\frac{B_l}{(2l)!}(4z)^{2l-1}\right) \\
&= \sum_{m=1}^{\infty}\left(i\frac{(2i)^m}{(m+1)!} - i\frac{(2i)^m}{m!} - 2\sum_{k+(2l-1)=m} 4^{2l-1}\frac{B_l}{(2l)!}\frac{(2i)^k}{k!}\right) z^m.
\end{aligned}$$

3.

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0}\left(\frac{2z}{z^2 - n^2}\right) = \frac{1}{z} - 2\pi\sum_{k=1}^{\infty}\frac{B_k}{(2k)!}(2\pi z)^{2k-1},$$

so that

$$\sum_{n \neq 0}\left(\frac{z}{n^2 - z^2}\right) = \pi\sum_{k=1}^{\infty}\frac{B_k}{(2k)!}(2\pi z)^{2k-1} = \frac{\pi^2 z}{6} + \frac{\pi^4 z^3}{90} + \frac{\pi^6 z^5}{3^2 \cdot 5 \cdot 7} + \dots.$$

But

$$\frac{1}{n^2 - z^2} = \frac{1}{n^2}\frac{1}{1 - (z/n)^2} = \frac{1}{n^2} + \frac{z^2}{n^4} + \frac{z^4}{n^6} + \dots$$

so that the LHS is

$$\sum_{n \in \mathbb{N}}\left(\frac{z}{n^2} + \frac{z^3}{n^4} + \frac{z^5}{n^6} + \dots\right) = \sum_{n \in \mathbb{N}}\frac{z}{n^2} + \sum_{n \in \mathbb{N}}\frac{z^3}{n^4} + \sum_{n \in \mathbb{N}}\frac{z^5}{n^6} + \dots.$$

Comparing the two Taylor series we must have

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n\in\mathbb{N}} \frac{1}{n^4} = \frac{\pi^4}{90} \qquad \text{and} \qquad \sum_{n\in\mathbb{N}} \frac{1}{n^6} = \frac{\pi^6}{3^2 \cdot 5 \cdot 7}.$$

4. We have

$$z^3 - n^3 = (z-n)(z-\omega n)(z-\omega^2 n),$$

where

$$\omega = e^{2\pi i/3}.$$

Using trial and error one can find the following partial fraction decomposition:

$$\frac{3z^2}{z^3-n^3} = \frac{1}{z-n} + \frac{\omega}{\omega z - n} + \frac{\omega^2}{\omega^2 z - n}.$$

Now we saw in class that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n\neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right),$$

As

$$1 + \omega + \omega^2 = 0,$$

we have

$$\begin{aligned}
\sum_{n\in\mathbb{N}} \frac{3z^2}{z^3-n^3} &= \sum_{n\in\mathbb{N}} \frac{1}{z-n} + \frac{\omega}{\omega z - n} + \frac{\omega^2}{\omega^2 z - n} \\
&= \frac{1}{z} + \sum_{n\in\mathbb{N}} \left(\frac{1}{z-n} + \frac{1}{n} \right) + \frac{\omega}{z} + \sum_{n\in\mathbb{N}} \left(\frac{\omega}{\omega z - n} + \frac{\omega}{n} \right) + \frac{\omega^2}{z} + \sum_{n\in\mathbb{N}} \left(\frac{\omega^2}{\omega^2 z - n} + \frac{\omega^2}{n} \right) \\
&= \pi \cot \pi z + \pi\omega \cot \pi\omega z + \pi\omega^2 \cot \pi\omega^2 z.
\end{aligned}$$

Therefore

$$\sum_{n\in\mathbb{N}} \frac{1}{z^3-n^3} = \frac{\pi}{3z^2} \cot \pi z + \frac{\pi\omega}{3z^2} \cot \pi\omega z + \frac{\pi\omega^2}{3z^2} \cot \pi\omega^2 z.$$