## MODEL ANSWERS TO THE SEVENTH HOMEWORK

1. Let $f(z) = 6z^3$ and $g(z) = z^7 - 2z^5 + 6z^3 - z + 1$. Then, $|f(z)| = 6$ on the circle $|z| = 1$ and

$$|g(z) - f(z)| \leq |z^7| + |2z^5| + |z| + |1| = 5 \qquad \text{on} \qquad |z| = 1.$$

Thus by Rouché, $f$ and $g$ have the same number of zeroes inside the unit disc. But $f(z)$ has a zero of order three at the origin and no other zeroes. So $g(z)$ has three zeroes in the unit disc.

2. On the circle $|z| = 2$, the dominant term is clearly $z^4$. Set $f(z) = z^4$ and $g(z) = z^4 - 6z + 3$. Then $|f(z)| = 2^4 = 16$ on the unit circle $|z| = 2$ and

$$|g(z) - f(z)| \leq |6z| + |3| \leq 15.$$

As $z^4$ has four zeroes inside the circle $|z| = 2$, by Rouché's Theorem so does $g(z)$.

On the other hand, on the circle $|z| = 1$ the dominant term is $6z$. Set $f(z) = 6z$. Then $|f(z)| = 6$ on the unit circle and

$$|g(z) - f(z)| \leq |z^4| + 3 = 4.$$

Thus, by Rouché $g(z)$ has one zero inside the unit circle. It follows that $g(z)$ has three zeroes in the annulus $1 \leq |z| \leq 2$.

3. (i) Note that as

$$\sin -x = -\sin x \qquad \text{and} \qquad \sin x = \sin(\pi - x)$$

we have

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{a + \sin^2 x} = \frac{1}{4} \int_0^{2\pi} \frac{\mathrm{d}x}{a + \sin^2 x}.$$

Now we use the well-known identity,

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

to get

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{a + \sin^2 x} = \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}x}{b - \cos 2x},$$

where $b = 2a + 1$.

Now put $z = e^{2ix}$ so that

$$\cos 2x = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{and} \qquad \mathrm{d}z = 2iz\mathrm{d}x.$$

Thus

$$\frac{1}{2}\int_0^{2\pi}\frac{dx}{b-\cos 2x} = -\frac{i}{2}\int_{|z|=1}\frac{1}{2z}\frac{dz}{b-\frac{1}{2}\left(z+\frac{1}{z}\right)}$$
$$= \frac{i}{2}\int_{|z|=1}\frac{dz}{z^2-2bz+1}.$$

We calculate the last integral using the Residue Theorem. The residues of $\frac{1}{z^2-2bz+1}$ are located at the zeroes of $z^2-2bz+1$. Using the quadratic formula, we see that the roots are

$$b \pm \sqrt{b^2-1} = 2a+1 \pm 2\sqrt{a^2+a}.$$

Call the positive root $\alpha$ and the negative root $\beta$. Since $|a| > 1$ the negative root $\beta$ is the only one inside the unit circle. It easy to calculate the residue at $\beta$,

$$\begin{aligned} R &= \lim_{z\to\beta}(z-\beta)f(z) \\ &= \lim_{z\to\beta}\frac{1}{z-\alpha} \\ &= \frac{1}{\beta-\alpha} \\ &= -\frac{1}{4(a^2+a)^{1/2}}. \end{aligned}$$

Thus

$$\int_0^{\pi/2}\frac{\mathrm{d}x}{a+\sin^2 x} = \frac{\pi}{4(a^2+a)^{1/2}}.$$

(ii) Let $\gamma$ be the contour that goes from 0 to $R$, along the real axis, describes the semi-circle from $R$ to $-R$ and then goes from $-R$ to 0 and consider integrating

$$f(z) = \frac{z^2}{z^4+5z^2+6},$$

over this contour. Using the method of partial fractions, we have

$$\frac{z^2}{z^4+5z+6} = \frac{-2}{z^2+2} + \frac{3}{z^2+3}.$$

Thus the poles of $f$ inside $\gamma$ are located at the points $z = \sqrt{2}i$ and $z = \sqrt{3}i$. These are both simple poles of $f$, so that we can compute the residues at these points as limits:

$$\lim_{z\to\sqrt{2}i}\frac{-2}{z+\sqrt{2}i} = \frac{i\sqrt{2}}{2},$$

and

$$\lim_{z\to\sqrt{3}i} \frac{3}{z+\sqrt{3}i} = \frac{-i\sqrt{3}}{2}.$$

Thus by the residue Theorem

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i(\frac{i\sqrt{2}}{2} + \frac{-i\sqrt{3}}{2}) = \pi(\sqrt{3}-\sqrt{2}).$$

Consider the integral around the semi-circle. We have

$$\left|\frac{z^2}{z^4+5z^2+6}\right| \leq \frac{R^2}{R^4-5R-6}.$$

Since the length of the semi-circle is $\pi R$ the integral around the semi-circle is easily seen to go to zero as $R$ goes to infinity. As $f(x)$ is even, taking the limit as $R\to\infty$, it follows

$$\int_0^\infty \frac{x^2\,\mathrm{d}x}{x^4+5x^2+6}\,\mathrm{d}x = \frac{\pi}{2}(\sqrt{3}-\sqrt{2}).$$

(iii) Let $\gamma$ be the same contour as above and let

$$f(z) = \frac{z^2}{(z^2+a^2)^3}.$$

The integral around the semi-circle is no more than

$$\frac{\pi R^3}{(R^2-a^2)^3}$$

which goes to zero as $R$ goes to infinity. As the integrand is an even function of $x$, the integral around $\gamma$ reduces, in the limit as $R$ tends to infinity, to twice the integral we are after. It suffices, then, to compute the residues of $f(z)$. Now the denominator is zero when $z=\pm ai$. Thus the only residue inside the contour is at $ai$. Unfortunately this is a pole of order three. Note that, in general, if $g(z)$ has a pole of order $k$ at $b$, then $h(z)=(z-b)^k g(z)$ is holomorphic and the residue of $g$ at $b$ is the coefficient of $(z-b)^{k-1}$ in $h(z)$. Thus the residue of $g$ at $b$ is

$$\lim_{z\to b} \frac{h^{(k-1)}(z)}{(k-1)!}.$$

For us, we should then look at

$$\frac{z^2}{(z+ia)^3}.$$

The first derivative is

$$\frac{z(2ia-z)}{(z+ia)^4},$$

and the second derivative is

$$\frac{2(ia-z)(z+ia)-4z(2ia-z)}{(z+ia)^5}.$$

Thus the residue at $z = ia$ is

$$\frac{1}{2}\frac{-4(ia)^2}{(2ia)^5} = -\frac{1}{2}\frac{i}{(2a)^3}.$$

Hence

$$\int_0^\infty \frac{x^2\,\mathrm{d}x}{(x^2+a^2)^3} = \frac{\pi}{2(2a)^3}.$$

(iv) Here we take a contour $\gamma$ that starts at $\rho$ and goes to $R$, along the $x$-axis, describes a semi-circle of radius $R$, counterclockwise, goes from $-R$ to $-\rho$ and then describes a semi-circle of radius $\rho$. Here we take

$$f(z) = \frac{\log(z)}{1+z^2}.$$

We choose a branch of the logarithm, so that the argument lies between $-\pi/2$ and $3\pi/2$ (so that we exclude the negative imaginary axis, $x = 0$, $y < 0$). On the circle $|z| = R$, we have

$$|f(z)| \leq \frac{(\log R + \pi)}{R^2 - 1}.$$

Thus the integral around the big circle is at most

$$2\pi R \frac{\log R + \pi}{R^2 - 1}$$

which tends to zero as $R \to \infty$. On the circle of radius $\rho$, the imaginary part is bounded (it lies between 0 and $\pi$) and the real part is $\log \rho$. As the length of the path is $2\pi\rho$, it follows that integral around the small semi-circle tends to zero, as $\rho \to 0$.

The only residue of $f(z)$ inside the circle is at $z = i$. The residue here is

$$\frac{\log i}{2i} = \frac{\pi}{4}.$$

As $f(x)$ is even, we have by the residue Theorem

$$2\int_0^\infty (1+x^2)^{-1} \log x \,\mathrm{d}x + \alpha = \frac{\pi^2 i}{2},$$

where $\alpha$ is the contribution from the two branches of the logarithm, so that it is purely imaginary.

As the integral is purely real, it follows that the integral is zero (and

$$\alpha = \frac{\pi^2 i}{2}.)$$