MODEL ANSWERS TO THE SIXTH HOMEWORK

1. As $f(z)$ is an entire holomorphic function it is analytic, so that it is given by a power series

$$f(z) = \sum_{m \in \mathbb{N}} a_m z^m,$$

with an infinite radius of convergence.
We apply Cauchy's inequality. Let γ be a circle of radius r. If r is sufficiently large then $|f(z)| \leq r^n$ so that

$$|a_m| \leq \frac{r^n}{r^m} = r^{n-m}.$$

As r tends to infinity this tends to zero for all $m > n$. Thus $a_m = 0$ for $m > n$. It follows that

$$f(z) = \sum_{m \leq n} a_m z^m,$$

so that $f(z)$ is a polynomial of degree at most n.
2. As $f(z)$ is an entire holomorphic function it is analytic, so that it is given by a power series

$$f(z) = \sum_{m \in \mathbb{N}} a_m z^m,$$

with an infinite radius of convergence.
Let

$$g(z) = f(1/z).$$

Then

$$g(z) = \sum_{m \in \mathbb{N}} a_m z^{-m},$$

valid for all $z \neq 0$.
By assumption we may find n such that $z^n g(z)$ has a removable singularity, so that it is given by a power series

$$z^n g(z) = \sum_{m \in \mathbb{N}} b_m z^m.$$

Comparing terms we must have $a_{n-m} = b_m$. Thus $a_m = 0$ for $m > n$ and $f(z)$ is a polynomial of degree at most n.

3. If $f(z)$ is a polynomial then some derivative $f^n(z)$ is the zero function.

$$\frac{d}{dz}e^z = e^z, \qquad \frac{d}{dz}\sin z = \cos z \qquad \text{and} \qquad \frac{d}{dz}\cos z = -\sin z.$$

Thus e^z, $\cos z$ and $\sin z$ are not polynomials. But then question 2 implies that they have essential singularities at ∞.

4. Let $f(z)$ be a meromorphic function. Since $\mathbb{P}^1$ is compact and the singularities of a meromorphic function are discrete, it follows that $f(z)$ has finitely many singularities. Suppose that f has poles in $\mathbb{C}$ at $a_1, a_2, \dots, a_k$ of orders $n_1, n_2, \dots, n_k$. Then

$$(z-a_1)^{n_1}(z-a_2)^{n_2}\dots(z-a_k)^{n_k}f(z)$$

has removable singularities so that it defines an entire function $g(z)$. As f has a pole at infinity so does $g(z)$. But then $g(z)$ is a polynomial by question 2. Thus

$$f(z) = \frac{g(z)}{(z-a_1)^{n_1}(z-a_2)^{n_2}\dots(z-a_k)^{n_k}}$$

is a rational function.

5. If we complete the square we get

$$f(z) = (z+1/2)^2 - 1/4.$$

Clearly this is the same question as finding where

$$g(z) = (z+1/2)^2$$

is injective. We just want to determine the radius of the biggest circle where g is injective.

This is the same as the largest radius of a circle centred at $1/2$ for which

$$h(z) = z^2$$

is injective.

h is two to one. It sends the point $z = re^{i\theta}$ and the point $z = re^{-i\theta}$ to the same point. It is clear that the largest circle about $1/2$ has radius $1/2$, since if the radius is more than half, we include points of this form.