MODEL ANSWERS TO THE FIFTH HOMEWORK

1. We try to find a map that sends the given region to an angular sector. To do this, we want to send both circles to straight lines through the origin. Thus we want to send one of the points of intersection of the two circles to the origin and the other point to the point at infinity.
So we first find where the two circles intersect. If we write $z = x + iy$, the two equations

$$\begin{aligned} |z| &= 1 \\ |z-1| &= 1 \end{aligned}$$

become

$$\begin{aligned} x^2 + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1. \end{aligned}$$

It follows that $x^2 = (x-1)^2$ so that $-2x + 1 = 0$ and $x = 1/2$ (we could have also seen this by the obvious symmetries). It follows that $y^2 = 3/4$ and $y = \pm\sqrt{3}/2$. Call the point above the y-axis α, so that the other point is $\bar{\alpha}$.
We choose to send the point above the y-axis to zero and the point below the y-axis to ∞. Such a map is given by

$$z \longrightarrow a\frac{z-\alpha}{z-\bar{\alpha}},$$

where a is any non-zero scalar. The point $z = 0$ is sent to $a\frac{\alpha}{\bar{\alpha}}$. It is natural to choose $a = \frac{\bar{\alpha}}{\alpha}$. In this case the circle $|z-1| = 1$ is sent to the x-axis.
To determine the image of the circle $|z| = 1$, note that the point $z = 1$ is sent to

$$\frac{\bar{\alpha}}{\alpha}\frac{1-\alpha}{1-\bar{\alpha}}.$$

After some routine algebra, we see that the circle $|z| = 1$ is sent to the line $\arg z = 2\pi/3$ (or indeed, holomorphic maps preserve angles and the angle between the two circles is $2\pi/3$ as the angle between the two radii is $\pi/3$).
To map this to the upper half plane, we want to increase the angle by a factor of $3/2$. The map $z \longrightarrow z^{3/2}$ will achieve this. Finally the map

$$z \longrightarrow \frac{z-i}{z+i},$$

will take the upper half plane to the interior of the unit disc
For what it is worth, the composition (obtained by multiplying two matrices together) will be

$$z \longrightarrow \frac{\bar{\alpha}}{\alpha} \frac{(1-i)z^{3/2} + i\bar{\alpha} - \alpha}{(1+i)z^{3/2} - \alpha - i\bar{\alpha}}.$$

Finally that as Möbius transformations preserve inverse points, to check that symmetries are preserved, it suffices to check that both symmetries with respect to the x-axis (the line $\arg z = 0$) and the line $\arg z = 2\pi/3$ are preserved by the map $z \longrightarrow z^{3/2}$. This is clear with respect the x-axis (complex conjugate points are taken to complex conjugate points). On the other hand, suppose that $z = re^{i\theta}$. Then the inverse point, with respect to the line $\arg z = 2\pi/3$, is the point $z^* = re^{i(4\pi/3-\theta)}$. Applying the map $z \longrightarrow z^3/2$ we get $w = r^{3/2}e^{3i\theta/2}$ and $w^* = r^{3/2}e^{i(2\pi-3\theta)} = r^{3/2}e^{-3i\theta/2}$, which is the complex conjugate of w.
Hence the given map preserves both types of symmetries.
2 (i). We have $\gamma(t) = (1+i)t$, for $t \in [0,1]$. Then $\gamma'(t) = 1+i$ and

$$\begin{aligned}
\int_\gamma x\,dz &= \int_0^1 x(\gamma(t))(1+i)\,dt \\
&= \int_0^1 t(1+i)\,dt \\
&= \left[t^2/2(1+i)\right]_0^1 \\
&= 1/2 + i/2.
\end{aligned}$$

(ii) We have $\gamma(t) = re^{it}$, for $t \in [0,2\pi]$. Then $\gamma'(t) = ire^{2\pi it}$ and

$$\begin{aligned}
\int_\gamma x\,dz &= \int_0^{2\pi} r\cos(t) rie^{it}\,dt \\
&= r^2 \int_0^{2\pi} \cos(t)e^{it}\,dt \\
&= r^2 \int_0^{2\pi} \cos^2(t) + i\cos(t)\sin(t)\,dt \\
&= r^2 \int_0^{2\pi} (1/2)1\,dt \\
&= r^2\pi i,
\end{aligned}$$

where we use periodicity to eliminate some of the integrals and some of the standard trigonometric identities.

Aliter:

$$\begin{aligned}\int_\gamma x\,dz &= \int_\gamma \frac{1}{2}(z+\frac{r^2}{z})\,dz\\ &= \frac{r^2}{2}\int_\gamma \frac{1}{z}\\ &= \pi i r^2,\end{aligned}$$

where we used the CIF.

(iii)

$$\begin{aligned}\int_\gamma \frac{dz}{z^2-1} = \frac{1}{2}\int_\gamma \frac{dz}{z-1} - \frac{1}{2}\int_\gamma \frac{dz}{z+1}\\ = \pi i 1 - \pi i 1 = 0,\end{aligned}$$

where we used partial fractions and the CIF.

(iv)

$$\begin{aligned}\int_\gamma \frac{e^z}{z^2-1}\,dz = \frac{1}{2}(\int_\gamma \frac{e^z}{z-1}\,dz) - \frac{1}{2}(\int_\gamma \frac{e^z}{z+1}\,dz)\\ = \pi i e + \pi i e^{-1},\end{aligned}$$

where we used partial fractions and the CIF.

(v)

$$\int_\gamma e^z z^{-n}\,dz = \frac{2\pi i}{(n-1)!},$$

using the version of Cauchy's Integral Formula that involves higher derivatives.

(vi)

$$\begin{aligned}\int_\gamma z^n(1-z)^{-m}\,dz &= \frac{2\pi i}{(m-1)!}f^{m-1}(1)\\ &= \frac{2\pi i}{(m-1)!}n(n-1)(n-2)\cdots(n-m+2),\\ &= \frac{2\pi i n!}{(m+1)!(m-1)!},\end{aligned}$$

using the version of Cauchy's Integral Formula that involves higher derivatives, where $f(z) = z^n$.