MODEL ANSWERS TO THE FOURTH HOMEWORK

1. Suppose the Möbius transformation is

$$z \longrightarrow \frac{az+b}{cz+d}.$$

Since $-i$ is sent to 0 we are reduced to

$$z \longrightarrow \frac{a(z+i)}{cz+d}.$$

As 0 is sent to 1, we must have

$$\frac{ia}{d} = 1 \qquad \text{so that} \qquad d = ai.$$

Thus we are reduced to

$$z \longrightarrow \frac{a(z+i)}{cz+ai}.$$

As i is sent to -1, we must have

$$\frac{2ai}{ci+ai} = -1 \qquad \text{so that} \qquad c = -3a.$$

Thus

$$z \longrightarrow \frac{(z+i)}{-3z+i}$$

does the trick.

2. We compute the cross-ratio of 1, -1, k and $-k$:

$$\lambda = \frac{-1+k}{-1-k}\frac{1-k}{1+k} = \left(\frac{k-1}{k+1}\right)^2$$

Consider the rational map

$$f\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \qquad \text{given by} \qquad z \longrightarrow \left(\frac{z-1}{z+1}\right)^2.$$

The degree of the rational function

$$\left(\frac{z-1}{z+1}\right)^2$$

is two. In particular f is surjective. Consider the equation

$$(z-1)^2 = \lambda(z+1)^2.$$

Rearranging we get:

$$(\lambda - 1)z^2 + 2(\lambda + 1)z + (\lambda - 1) = 0.$$

The discriminant is

$$4(\lambda + 1)^2 - 4(\lambda - 1)^2.$$

There are two solutions unless the discriminant is zero, that is, unless

$$(\lambda + 1)^2 = (\lambda - 1)^2.$$

Thus there are two solutions unless $\lambda = 1/2$.
Given four points distinct points in $\mathbb{P}^1$ we can find a linear transformation carrying those four points to 1, -1, k and $-k$ if and only if the cross-ratio of the points is

$$\left(\frac{k-1}{k+1}\right)^2.$$

There are two possible values of k unless the cross-ratio is $1/2$.
3. We want to write down a Steiner system such that these two circles belong to one family of the Steiner system; the circles from the other Steiner system are orthogonal to both circles. The two families from a Steiner system are circles containing two points and circles of Apollonius. The two given circles don't have any points in common, so we want to realise the two circles as circles of Apollonius.
We want to find two points a and b such that a and b are simultaneously inverse points for both circles. Now such a pair of points must lie on the line connecting their centres. Since both centres lie on the x-axis, the points we are looking for must lie on the x-axis.
Suppose that one of the points is x. As the centre of one of the circles is 0, and the radius is one, it follows that the other point is $1/x$. On the other hand, the inverse x' of the point x in the circle with centre 1 is given by the equation

$$(x' - 1)(x - 1) = 4^2 = 16.$$

Thus we get the quadratic equation

$$(1/x - 1)(x - 1) = 16.$$

Solving this we get

$$(1 - x)(x - 1) = 16x \qquad \text{so that} \qquad x^2 + 14x + 1 = 0.$$

Using the quadratic formula, we get two solutions

$$x = -7 \pm \sqrt{49 - 1}.$$

Suppose that the solutions are a and b. The circles we are looking for are the circles that contain both a and b. The equation of these circles is

$$\arg(z-a) = \arg(z-b) + \phi + l\pi,$$

where ϕ is fixed and $l \in \{-1, 0, 1\}$. Varying ϕ gives the family of orthogonal circles.

4. Note that the origin and the point at infinity are inverse points with respect to the unit circle. Suppose that we send the point $\alpha = re^{i\theta}$ to the origin. Then the inverse point $\alpha^* = 1/re^{i\theta} = 1/\bar{\alpha}$ must be sent to infinity.

Thus our Möbius transformation must have the form

$$z \longrightarrow a\frac{z-\alpha}{z-\alpha^*} = \lambda\frac{z-\alpha}{\bar{\alpha}z-1},$$

for some complex number λ. As 1 is mapped to a point of the unit circle, we get that $|\lambda| = 1$, so that we may write our map as

$$z \longrightarrow w = e^{i\phi}\frac{z-\alpha}{\bar{\alpha}z-1},$$

where $|\alpha| < 1$. It is easy to check that $|w| < 1$ for $|z| < 1$.

5. The map $z \longrightarrow z^2$ sends the region $\operatorname{Im} z > 0$ and $\operatorname{Re} z > 0$ to the upper half plane, $\operatorname{Im} z > 0$. We have already seen that the map

$$z \longrightarrow \frac{z-i}{z+i}$$

sends the upper half plane to the unit disc. So the composition

$$z \longrightarrow \frac{z^2-i}{z^2+i}$$

will do the trick.