MODEL ANSWERS TO THE SECOND HOMEWORK

1. (i) Suppose that (a, b, c) is a point of the sphere. The antipodal point is $(-a, -b, -c)$. $(-a, -b, -c)$ is sent to the
Thus antipodal points are sent to points z, z' such that $z\bar{z}' = -1$. Given a non-zero complex number z there is a unique complex number $\bar{z}'$ such that $z\bar{z}' = -1$; indeed

$$z' = -\frac{1}{\bar{z}}.$$

On the other hand, given any point of the sphere there is a unique antipodal point.
(ii) The points on the top face all lie on horizontal circle. This circle is sent to a circle centred at the oigin in $\mathbb{C}^2$. The four vertices are sent to the four points on this circle which make an angle of $\pi/4$ with the real and imaginary axes.
It suffices to calculate the radius r of this circle. We use similar triangles. Consider the distance from a vertex to the z-axis. This is the hypotenuse of a right-angled triangle whose other sides are $1/2$. Thus this distance is

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}.$$

The same line segment is the side of a right-angled triangle whose hypotenuse is goes from the centre of the triangle to a vertex which has length 1. Thus the other side has length

$$\sqrt{1^2 - \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}}.$$

One triangle of the two similar triangles has a vertex at the North pole, a vertex on the z axis and the top of the cube and the third vertex a vertex of the cube on the same face. The other triangle has vertices the North pole, the origin and the projection of this vertex of the cube. We have

$$\frac{r}{1/2} = \frac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}}.$$

Thus

$$r = \frac{1}{2(\sqrt{2} - 1)}.$$

It follows that the four vertices on the top of the cube are sent to

$$\frac{2+\sqrt{2}}{4}(1+i), \quad \frac{2+\sqrt{2}}{4}(-1+i), \quad \frac{2+\sqrt{2}}{4}(-1-i), \quad \frac{2+\sqrt{2}}{4}(1-i).$$

The other four vertices of the cube are antipodal to the first four points. By (i) the other four points are

$$\frac{2}{2+\sqrt{2}}(1+i), \quad \frac{2}{2+\sqrt{2}}(-1+i), \quad \frac{2}{2+\sqrt{2}}(-1-i), \quad \frac{2}{2+\sqrt{2}}(1-i).$$

2. We have to show that there is a unique Möbius transformation that carries three distinct point p_1, p_2, p_3 to three other distinct points q_1, q_2, q_3.
To show that the group of Möbius transformations is thrice transitive it suffices to prove that the orbit of any set of three points, is the set of all three-tuples.
In other words we are free to choose one set of points as we please. We choose $q_1 = \infty$, $q_2 = 0$ and $q_3 = 1$.
We first send p_1 to q_1. We may assume that $p_1 \neq \infty$. Then $p_1 = a \in \mathbb{C}$ and $z \longrightarrow 1/(z-a)$ sends p_1 to q_1. Thus we may assume $p_1 = q_1 = \infty$. Now $z \longrightarrow az + b$ fixes p_1. Suppose that $p_2 = \lambda \in \mathbb{C}$. Then choose $a = 1$ and $b = -\lambda$. This sends p_2 to $q_2 = 0$.
Thus we may assume $p_1 = q_1$ and $p_2 = q_2$. Note that $z \longrightarrow az$ fixes p_1 and p_2.
Finally suppose that $p_3 = \mu$. The transformation $z \longrightarrow (1/\mu)z$ maps p_3 to q_3.
Thus the group of Möbius transformations is certainly thrice transitive.
Now suppose that the Möbius transformation

$$z \longrightarrow \frac{az+b}{cz+d}$$

fixes 0, 1 and ∞. As it fixes ∞, $c = 0$. Rescaling we may assume $d = 1$. As it fixes 0, $b = 0$. Finally as it fixes 1, $a = 1$. But then we have the identity transformation.
3. There are two ways to do this problem. Both start the same way. Note that if we differentiate $f(z)$ then we get $\sum n a_n z^{n-1}$.
The most straightforward thing to do is then multiply this by z to get $zf' = \sum n a_n z^n$. Repeating this three times gives us what we want

$$(z(z(zf)')')' = \sum n^3 a_n z^n.$$

Another way to proceed is to differentiate f three times. This gives

$$\sum n(n-1)(n-2) a_n z^{n-3}.$$

Now multiply by z^3 to get

$$\sum n(n-1)(n-2)a_n z^n.$$

This is almost right. Subtracting, we see that we get an error involving a quadratic polynomial in n. Differentiating twice and doing the same thing, we reduce the error to a linear polynomial and so on. Details are left to the reader.

4. Easy, take $z_{2n} = n$, $z_{2n+1} = 0$. This sequence is unbounded but has no limit.

5. The sequence z_n converges to infinity if and only if the sequence $y_n = 1/z_n$ tends to zero. Now nothing can be said about the real or imaginary parts of z_n, apart from the fact that at least one must be tending to infinity. Indeed the sequence $z_n = a + ibn$, where a and b are reals and a is arbitrary and b is non-zero, tends to infinity. On the other hand the real part is constant.

Replacing z_n by iz_n, clearly the same holds for the imaginary part. Given any complex number z, the sequence $z_n = nz$ tends to infinity. Taking $z = e^{i\theta}$ it is clear that the argument of the sequence tends to θ. On the other hand the modulus $|z|$ has to approach infinity.