MODEL ANSWERS TO THE FIRST HOMEWORK

1. It suffices to prove the following general result:

Lemma 0.1. *Let $f(x)$ and $g(x)$ be general real rational functions. Let n be a positive integer.*

(1) Then the nth derivative of

$$f(x)\exp(g(x)),$$

in a punctured neighbourhood of zero, has the form

$$f_1(x)\exp(g(x)),$$

where $f_1(x)$ is a rational function.

Now suppose that $g(x) = -1/x^2$.

(2) The limit of

$$f(x)\exp(g(x))$$

as x approaches zero is zero.

(3) Define a function $\phi(x)$ in a neighbourhood of zero, by setting

$$\phi(x) = \begin{cases} f(x)\exp(g(x)) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases}$$

where $g(x) = 1/x^2$. Then the nth derivative of $\phi(x)$ at zero is zero.

Proof. We first prove (1). By an obvious induction it suffices to prove this result in the case $n = 1$. Since we are working in a punctured neighbourhood of zero, we may assume that both $f(x)$ and $g(x)$ are defined. We can then calculate the derivative of $f(x)\exp(g(x))$ using the standard rules of calculus. We get

$$\begin{aligned}[f(x)\exp(g(x))]' &= f'(x)\exp(g(x)) + f(x)g'(x)\exp(g(x)) \\ &= f_1(x)\exp(g(x)),\end{aligned}$$

where $f_1(x)$ is the rational function $f'(x) + f(x)g'(x)$. Hence (1).
To prove (2), note that we may write $f(x) = x^n f_1(x)$, where both the numerator and denonominator of $f_1(x)$ are coprime to x. Since $\lim_{x\to 0} f_1(x) = f_1(0)$ it suffices to show that

$$\lim_{x\to 0} x^n \exp(g(x))$$

is equal to zero. The first trick is to replace x by $y = 1/x^2$. As x approaches zero, y approaches ∞. Thus we are reduced to calculating

$$\lim_{x\to\infty} x^{n/2}\exp(-x).$$

Now if $n < 0$ the limit is obviously zero, since both terms are approaching zero. Otherwise one term is going to zero and the other to infinity. Applying L' Hôpital's rule enough times, we reduce to the case $n < 0$ (in fact $n = -1/2$) and the limit is zero. Hence (2).

By (1) and induction, we may assume that $n = 1$. We need to calculate the limit

$$\lim_{x\to 0} \frac{f(x)\exp(g(x)) - \phi(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)\exp(g(x))}{x}.$$

Now apply (2). $\square$

2. Same as 1.
3. We have

$$u(x, y) = \frac{x^2y^2}{x^2 + y^4},$$

and

$$v(x, y) = \frac{xy^3}{x^2 + y^4}.$$

Consider first what happens as we approach zero along a line. Take the line $y = mx$ and suppose that $x \neq 0$. Then

$$u(x, y) = u(x, mx) = m^2\frac{x^4}{x^2 + m^4x^4} = m^2\frac{x^2}{1 + m^4x^2}.$$

Thus the limit as x approaches zero is zero. Moreover if $m = 0$, then we get $u(x, 0) = 0$. If we look at $x = 0$, then we get $u(0, y) = 0$.

In terms of v we get

$$v(x, y) = v(x, mx) = m^3\frac{x^4}{x^2 + m^4x^4} = m^3\frac{x^2}{1 + m^4x^2}.$$

Thus the limit as x approaches zero is zero. Moreover if $m = 0$, then we get $v(x, 0) = 0$. If we look at $x = 0$, then we get $v(0, y) = 0$. In particular

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial u}{\partial y} = 0, \qquad \text{and} \qquad \frac{\partial v}{\partial y} = 0,$$

at zero, and so the Cauchy-Riemann equations are trivially satisfied. Now suppose that you take the family of conics $x = my^2$. Then

$$f(x, y) = f(my^2, y) = \frac{my^4(my^2 + iy)}{m^2y^4 + y^4} = my\frac{y + i}{1 + m^2}.$$

In this case
$$\frac{\partial f}{\partial y} = m\frac{2y+i}{1+m^2}.$$
Evaluating at $y = 0$ we get
$$\frac{\partial f}{\partial y} = i\frac{m}{1+m^2}.$$
Clearly this depends on the choice of m. For example if $m = 0$ we get zero, but if $m = 1$ we get $i/2$. Thus f is not differentiable, since the value of the limit depends on the path you choose to approach the origin.

Note that we did not show that u and v are $\mathcal{C}^1$, just that their derivatives exist. The proposition proved in class requires this condition.

4. There are two ways to prove this. Essentially we prove an appropriate chain rule in both cases.

Suppose that z is a point where g is holomorphic and f is holomorphic at $g(z)$. We may suppose that $z = 0$. We show that
$$\lim_{z\to 0} \frac{f(g(z)) - f(g(0))}{z - 0} = f'(g(z))g'(z).$$
There are two cases. Suppose that there is a solution to $g(z) = g(0)$, in any punctured neighbourhood of 0. We claim that both sides are zero. As g is differentiable at 0, it follows that $g'(0) = 0$. Thus the RHS is zero. On other the hand, if $g(z) = g(0)$ then the numerator of the limit is zero and so the LHS is zero.

Thus to compute the limit, we may assume that $g(z) \neq g(0)$. In this case
$$\frac{f(g(z)) - f(g(0))}{z - 0} = \frac{f(g(z) - f(g(0))}{g(z) - g(0)}\frac{g(z) - g(0)}{z - 0}.$$
Since the limit of a product is the product of the limits, we are done.

Aliter: Suppose that $w = g(z)$ and that f and g are arbitrary $\mathcal{C}^1$ functions of x and y. Note that there is a chain rule for $\frac{\partial}{\partial \bar{z}}$
$$\frac{\partial f(g(z))}{\partial \bar{z}} = \frac{\partial f}{\partial w}\frac{\partial g}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{w}}\frac{\partial \bar{g}}{\partial \bar{z}},$$
which can be proved formally, from the definition of the operator $\frac{\partial}{\partial \bar{z}}$.
But if f and g are holomorphic, then
$$\frac{\partial g}{\partial \bar{z}} = 0,$$
and
$$\frac{\partial f}{\partial \bar{w}} = 0,$$

and so

$$\frac{\partial f(g(z))}{\partial \bar{z}} = 0$$

and we are done.

5. Set $u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$. For u to be harmonic we must have

$$\nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We get

$$6ax + 2by + 2cx + 6dy = 0.$$

Equating coefficients of x and y gives

$$c = -3a \qquad \text{and} \qquad b = -3d.$$

Thus $u(x,y) = ax^3 - 3dx^2y - 3axy^2 + dy^3$ is the general form. In this case a harmonic conjugate would satisfy the differential equations

$$\frac{\partial v}{\partial y} = 3ax^2 - 6dxy - 3ay^2 \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 3dx^2 + 6axy - 3dy^2.$$

Integrating the first equation gives

$$v(x,y) = 3ax^2y - 3dxy^2 - ay^3 + \phi(x),$$

where $\phi(x)$ is an arbitrary function of x. Plugging this into the second equation gives

$$\phi'(x) + 6axy - 3dy^2 = 3dx^2 + 6axy - 3dy^2.$$

Thus

$$\phi'(x) = 3dx^2,$$

and so $\phi(x) = dx^3$ is a solution. Thus the harmonic conjugate is

$$v(x,y) = dx^3 + 3ax^2y - 3dxy^2 - ay^3.$$

Aliter: One can also use the trick introduced in the 2nd lecture. If $f(z) = u + iv$ then

$$\begin{aligned} f(z) &= 2u(z/2, z/2i) - u(0,0) \\ &= 2(a(z/2)^3 - 3d(z/2)^2(z/2i) - 3a(z/2)(z/2i)^2 + d(z/2i)^3) \\ &= (a + 3di + 3a + di)z^3/4 \\ &= (a + di)z^3 \\ &= (a + di)(x + iy)^3. \end{aligned}$$

The imaginary part is

$$v(x,y) = dx^3 + 3ax^2y - 3dxy^2 - ay^3.$$