MIDTERM
MATH 220A, UCSD, AUTUMN 14

You have 50 minutes.

There are 5 problems, and the total number of points is 70. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	10	
6	10	
7	10	
Total	70	

1. (15pts) Write down the Cauchy-Riemann equations (the complex or real form as you wish) and show that any holomorphic function must satisfy them.
Under what conditions is it true that a function which satisfies the Cauchy-Riemann equations is holomorphic?

Solution:

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Suppose that f is holomorphic. Then f is differentiable and the following limit exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Take $z = x + iy_0$. Then we get

$$\lim x \to x_0 \frac{f(x + iy_0) - f(x_0 + iy_0)}{x - x_0} = \frac{\partial f}{\partial x}.$$

Now take $z = x_0 + iy$. Then we get

$$\lim y \to y_0 \frac{f(x_0 + iy) - f(x_0 + iy_0)}{i(y - y_0)} = -i\frac{\partial f}{\partial y}.$$

These must be equal for the limit to exist.
If the partial derivatives are continuous and satisfy the Cauchy-Riemann equations, then f is holomorphic.

2. (15pts) Find a power series expansion for

$$\frac{2z-1}{z-3}$$

about the point $z = 2$. What is the radius of convergence?

Solution:

$$\begin{aligned}\frac{2z-1}{z-3} &= \frac{2z-6}{z-3} + 5\frac{1}{z-3} \\ &= 2 - 5\frac{1}{1-(z-2)} \\ &= 2 - 5\left(1 + (z-2) + (z-2)^2 + (z-2)^3 + \dots\right) \\ &= -3 - 5(z-2) - 5(z-2)^2 - 5(z-2)^3 + \dots\end{aligned}$$

The radius of convergence is one.

3. (15pts) Find a conformal transformation of the region $0 < \operatorname{Re} z < 1$ onto the interior of the unit disc.

Solution:
First rotate by ninety degrees,

$$z \longrightarrow e^{i\pi/2} z,$$

to get the strip $0 < \operatorname{Im} z < 1$. Now multiply by π,

$$z \longrightarrow \pi z,$$

to get the strip $0 < \operatorname{Im} z < \pi$. Now take the exponential,

$$z \longrightarrow e^z,$$

to get the angular sector $0 < \theta < \pi$, that is, the upper half plane. Now map the upper half plane to the unit circle, using one of the standard maps, for example,

$$z \longrightarrow \frac{z - i}{z + i}.$$

4. (15pts) State a version of Cauchy's Theorem for rectangles, that involves functions which are holomorphic except possibly at finitely many points, and derive Cauchy's Integral Formula, for a rectangle, from this version.

Solution: Let $f(z)$ be a holomorphic inside a region U that contains a rectangle R, with a finite number of points $a_1, a_2, \dots, a_k$ such that

$$\lim_{z \to a_i} (z - a_i) f(z) = 0.$$

Let γ be the boundary of the rectangle. Then

$$\int_\gamma f(z) \, \mathrm{d}z = 0.$$

Now let us derive Cauchy's Integral formula. Consider the function

$$g(z) = \frac{f(z) - f(a)}{z - a},$$

where a is a point inside the rectangle. Then

$$\lim_{z \to a} (z - a) g(z) = 0.$$

Thus

$$\int_\gamma g(z) \, \mathrm{d}z = 0.$$

On the other hand we claim that the winding number

$$\frac{1}{2\pi i} \int_\gamma \frac{1}{z - a} \, dz = 1$$

There are manys ways to proceed. For example, $\mathbb{C} - \gamma$ has two components and the winding number is constant on the components. So we may assume that a is at the centre of the rectangle. Now one can proceed by direct computation. One can also argue that the this is the same as the winding number of a small circle centred at a.
Once the claim is established, the result follows easily.

5. (10pts) Evaluate the integral

$$\int_\gamma \frac{\sin z}{z^n}\,\mathrm{d}z,$$

where γ is a circle that contains the origin as an interior point.

Solution:
If $n \geq 0$ the integral is zero by Cauchy's Theorem. So we may assume that $n \leq -1$. Let $f(z) = \sin z$. Then by Cauchy's Integral Formula,

$$f^{(n-1)}(0) = \frac{(n-1)!}{2\pi i} \int_\gamma \frac{f(z)}{z^n}\,\mathrm{d}z.$$

Now $f(z) = \sin z$ so that

$$f^{(n-1)}(z) = \begin{cases} \cos z & \text{if } n \equiv 2 \mod 4 \\ -\sin z & \text{if } n \equiv 3 \mod 4 \\ -\cos z & \text{if } n \equiv 0 \mod 4 \\ \sin z & \text{if } n \equiv 1 \mod 4. \end{cases}$$

Therefore

$$f^{(n-1)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \equiv 2 \mod 4 \\ -1 & \text{if } n \equiv 0 \mod 4. \end{cases}$$

Thus

$$\int_\gamma \frac{\sin z}{z^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2\pi i}{(n-1)!} & \text{if } n \equiv 2 \mod 4 \\ -\frac{2\pi i}{(n-1)!} & \text{if } n \equiv 0 \mod 4. \end{cases}$$

Bonus Challenge Problems

6. (10pts) Classify all Möbius Transformations that send the upper half plane to the upper half plane.

Solution: We use the classification of all Möbius Transformations of the upper half plane to the unit disc. These are given as

$$z \longrightarrow e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}},$$

where λ is real and $\operatorname{Im}\alpha > 0$.

Now compose this, with any map back to the upper half plane, for example the inverse of

$$z \longrightarrow \frac{z-i}{z+i},$$

which is

$$z \longrightarrow -i\frac{z+1}{z-1}$$

Thus we get

$$z \longrightarrow e^{i\lambda}\frac{-(i+1)z+\alpha-1}{-(i+\bar{\alpha})z-i+\bar{\alpha}}.$$

7. (10pts) Is it true that a function which satisfies the Cauchy-Riemann equations is holomorphic?

Solution: No, we need that f has continuous partial derivatives. Look at the first hwk problem set.