8. More about Möbius Transformations

Recall that a Möbius Transformations is a rational function of degree one, so that as a transformation f of the extended complex plane

$$z \longrightarrow \frac{az+b}{cz+d}.$$

Here $f(\infty) = a/c$ and $f(-d/c) = \infty$. Möbius Transformations are sometimes called linear transformations, for obvious reasons.

It is not hard to prove that the group of Möbius Transformations is exactly thrice transitive.

Indeed if z_1, z_2, z_3 are three points of $\mathbb{C}$, then

$$z \longrightarrow \frac{z_3 - z_2}{z_3 - z_1} \frac{z - z_1}{z - z_2}$$

sends z_1 to zero, z_2 to infinity and z_3 to 1.

On the other hand, by direct computation, it is not hard to show that the only Möbius Transformation which fixes 0, 1 and ∞ is the identity transformation.

Using this fact, one can define an extremely important invariant of four ordered points z_1, z_2, z_3 and z_4 in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

Definition 8.1. *Let z_1, z_2, z_3 and z_4 be four distinct points of the extended complex plane.*

*The **cross-ratio** of these four points is*

$$\frac{z_2 - z_4}{z_2 - z_3} \frac{z_1 - z_3}{z_1 - z_4}.$$

By what we have just said, the cross-ratio λ is the image of z_1 when you use a Möbius transformation to send z_2 to 1, z_3 to 0 and z_4 to ∞. Note the cross-ratio is invariant under the action of the Möbius Transformations and vice-versa:

Lemma 8.2. *Given a pair of four distinct ordered points z_1, z_2, z_3 and z_4, and z'_1, z'_2, z'_3 and z'_4 of the extended complex plane, we may find a Möbius Transformations carrying one set to the other if and only if the cross-ratios λ and λ' are equal.*

On the other hand the cross-ratio is not an invariant under changing the order:

Lemma 8.3. *The group of Möbius Transformations that preserve the set $\{0, 1, \infty\}$ is isomorphic to S_3, generated by*

$$z \longrightarrow 1/z \qquad \textit{and} \qquad z \longrightarrow 1 - z.$$

Proof. Since the group of Möbius Transformations is precisely thrice transitive, the subgroup of Möbius Transformations that preserve the set $\{0, 1, \infty\}$ is isomorphic to S_3.

On the other hand

$$z \longrightarrow 1/z \qquad \text{and} \qquad z \longrightarrow 1 - z$$

map to two different transpositions, and any two transpositions generate S_3. $\square$

The stabiliser of the set $\{z_1, z_2, z_3, z_4\}$ turns out to always contain the Klein VierrerGruppe V. Indeed the transformation,

$$z \longrightarrow \lambda/z$$

switches 0 and ∞ and 1 and λ, and by symmetry we can get any other permutation which switches two pairs of elements. The quotient of S_4 by the subgroup V is S_3.

Definition-Lemma 8.4. *Given four points z_1, z_2, z_3 and z_4 in the plane, the j-invariant is the number*

$$2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2},$$

where λ is the cross-ratio.

Two sets of four points in $\mathbb{P}^1$ may be mapped to each other by an Möbius transformation of $\mathbb{P}^1$ if and only if they have the same j-invariant.

Proof. We first check that the j-invariant does not depend on the order.

It suffices to check what happens under the two transpositions. This is an easy check.

Now use a little Galois theory. We want the fixed field K of S_3 acting on $L = \mathbb{C}(\lambda)$. Then L/K is Galois, and the degree is six.

Now $E = \mathbb{C}(j) \subset K$ and λ satisfies a degree six polynomial over E. Thus L/E has degree at most six. But then $K = E$. $\square$

It is interesting to note that there are two very special values of the j-invariant, 0 and 1728. They correspond to the sets of four points which have extra automorphisms. In fact, one set corresponds to $0, 1, \infty, -1$. Here we have an extra $\mathbb{Z}_2$, corresponding to

$$z \longrightarrow 1/z.$$

The other corresponds to $1, \omega, \omega^2, 0$, where ω is a cube root of unity. Here we have an extra $\mathbb{Z}_3$, corresponding to

$$z \longrightarrow \omega z.$$