6. Examples of Functions defined by Series

We look at some interesting examples of functions given by power series. Consider the differential equation

$$y'(z) = y,$$

subject to the initial value $y(0) = 1$. We look for solutions y which are holomorphic functions of z.

We posit a solution that is given by a power series with centre the origin,

$$y(z) = \sum a_n z^n.$$

Then

$$y'(z) = \sum (n+1) a_n z^n \qquad \text{and} \qquad y(0) = a_0.$$

Hence the initial condition implies that

$$a_0 = 1.$$

As $y'(z) = y(z)$, comparing terms, we get

$$a_{n+1} = a_n/(n+1).$$

Clearly the unique solution to this recurrence relation is

$$a_n = 1/n!.$$

Thus we get

$$y(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

For obvious reasons we call this function the exponential function. Note that

$$\liminf (n!)^{1/n} \geq \liminf (n/2)^{1/2} = \infty,$$

(since we are taking reciprocals the limsup gets replaced by a liminf) so that the radius of convergence is infinity, that is, the exponential function is everywhere holomorphic, that is, the exponential function is entire.

Note that the holomorphic function $f(z) = e^{a+z}$ satsifies the differential equation

$$f' = f,$$

subject to the initial condition $f(0) = e^a$. On the other hand this differential equation has the unique solution $f(z) = e^a e^z$. Thus

$$e^{a+b} = e^a e^b,$$

for all complex numbers a and b.

In particular $e^z e^{-z} = e^0 = 1$ and so e^z is never zero. As the coefficients of the power series are all real

$$e^{\bar{z}} = \overline{e^z}.$$

So

$$|e^{iy}|^2 = e^{iy}e^{-iy} = e^0 = 1$$

and

$$|e^{x+iy}| = |e^x|.$$

Having defined e^z, it is possible to define two other entire holomorphic functions,

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2},$$

and

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.$$

Then

$$\cos(z) = 1 - z^2/2 + z^4/4! + \dots$$

and

$$\sin(z) = z - z^3/3! + z^5/5! + \dots.$$

By definition

$$e^{iz} = \cos z + i \sin z,$$

and so

$$\cos^2 z + \sin^2 z = 1.$$

Consider the periodicity of e^{iz}. Suppose that

$$e^{i(z+c)} = e^{iz}.$$

Then $e^{ic} = 1$. Since 1 is a point on the unit circle, ic must be imaginary, that is, $c = \theta \in \mathbb{R}$, where $e^{i\theta} = 1$. Using standard arguments, one can show that there is a non-zero real number θ such that $e^{i\theta} = 1$.

On the other hand, consider the map

$$f \colon \mathbb{R} \longrightarrow S^1 \qquad \text{given by} \qquad c \longrightarrow e^{ic},$$

where S^1 is the unit circle $|z| = 1$. f is a homomorphism of topological groups, that is, f is a group homomorphism of the additive group to the circle and f is continuous. The kernel is a closed subgroup.

Proposition 6.1. *Let $f \colon U \longrightarrow \mathbb{C}$ be a holomorphic function.*

Then f is constant if f' is zero, or the real part u is constant, or the imaginary part v is constant, or the modulus is constant, or the argument is constant.

Proof. If $f' = 0$ then all of the partials are zero and both u and v are constant.

Suppose that u is constant. Then

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 0,$$

and so f is constant. If v is constant then the real part of the holomorphic function if is constant and so f is constant.

Suppose the modulus is constant. Then $u^2 + v^2 = 0$ is constant and so

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0.$$

Similarly

$$0 = u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = -u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}.$$

These two simultaneous linear equations imply that either

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0,$$

or that the determinant $u^2+v^2 = 0$. In the latter case $f = 0$ is constant. Either way f is constant.

Finally if the argument is constant then $u = kv$ for some constant k (or v is identically zero, in which case f is constant). But $u - kv$ is the real part of $(1 + ik)f$ and so f must be constant. □

By (6.1) applied to the entire holomorphic function $z \longrightarrow e^{iz}$, the kernel is not the whole of $\mathbb{R}$, since then the argument of e^{iz} is constant and so e^{iz} is a constant function.

Since the kernel is closed there must be a smallest such θ. This is called the period and it is denoted by 2π. Clearly this definition of π is consistent with the standard one.

We want to define the logarithm $\log(z)$ of z. Clearly the logarithm should be the inverse of the exponential. That is, if

$$w = \log(z) \qquad \text{then} \qquad z = e^w.$$

Unfortunately the inverse is not uniquely defined, simply because the exponential is periodic, so that there are infinitely many w such that $z = e^w$. If w_0 is one of them, then they are all given by $w_0 + 2k\pi i$, where $k \in \mathbb{Z}$ is an integer.

A **region** U is any connected open subset of $\mathbb{C}$. A **branch** of the logarithm on U, is a continuous function $w = f(z) = \log(z)$ on U, such that $e^w = z$. Given one branch $f(z)$ there are infinitely many others, given by $f(z) + 2k\pi i$, where $k \in \mathbb{Z}$ is an integer.

We will show that we can construct a branch of the logarithm, on the region

$$U = \mathbb{C} - \{ z \in \mathbb{C} \mid \operatorname{Im}(z) = 0, \operatorname{Re}(z) \leq 0 \}.$$

Suppose that $w = x + iy$. Then the equation

$$e^w = z,$$

reduces to the two equations,

$$e^x = |z|$$

and

$$e^{iy} = \frac{z}{|z|}.$$

The first equation has the solution $x = \log(|z|)$, where we take the ordinary real logarithm. The second equation has infinitely many solutions. We pick the unique solution such that $-\pi < y < \pi$.

It is convenient to rewrite all of this in the form

$$z = re^{i\theta}.$$

Here

$$\theta = \log(\frac{z}{|z|})$$

and

$$r = |z|.$$

θ is called the argument, denoted $\arg z$, and $|z|$ is called the modulus. We check that this choice of θ gives us a continuous function for the logarithm.

Suppose $w_1 = u_1 + iv_1$, where $|v_1| < \pi$. Fix $\epsilon > 0$. Consider the subset A of $\mathbb{C}$ given by

$$|w - w_1| \geq \epsilon, \qquad |v_1| < \pi \qquad \text{and} \qquad |u - u_1| < \log 2.$$

This is closed and bounded, and so it is compact, and it is non-empty, if ϵ is sufficiently small. The function

$$|e^w - e^{w_1}|$$

is continuous and so it attains its minimum ρ. $\rho > 0$ as A does not contain any point of the form

$$w_1 + 2k\pi i.$$

Let

$$\delta = \min(\rho, \frac{1}{2}e^{u_1}).$$

Suppose that

$$|z - z_1| = |e^w - e^{w_1}| < \delta.$$

Then $w \notin A$ by choice of ρ. If $u < u_1 - \log 2$ then

$$|e^w - e^{w_1}| \geq e^{u_1} - e^u > \frac{1}{2}e^{u_1} > \delta$$

impossible and if $u > u_1 - \log 2$ then

$$|e^w - e^{w_1}| \geq e^u - e^{u_1} > e^{u_1} > \delta$$

impossible.

Thus $|w - w_1| < \epsilon$ and the function is continuous. It is easy to see that the logarithm is a holomorphic function, whose derivative is $1/z$. This is essentially the inverse function theorem. Having chosen a branch of the logarithm, we get branches of other well-known functions.

For example, consider defining a branch of the square root $w = f(z) = \sqrt{z}$. We define the branch on the same open subset. We want to solve

$$w^2 = z.$$

Taking logs of both sides, we get

$$2\log(w) = \log(z).$$

Thus

$$w = \exp(\log z/2).$$

If we write $z = re^{i\theta}$, then

$$\log(z) = \log(r) + i\theta.$$

So

$$\log z/2 = \log r^{1/2} + i\theta/2,$$

and

$$w = \sqrt{r}e^{i\theta/2}.$$

That is, to find the square root on this branch, simply take the square root of the modulus and half of the angle. With this choice of branch,

$$\sqrt{i} = \exp(i\pi/4) = \frac{1}{\sqrt{2}}(1 + i).$$

Of course the other solution to the equation

$$z^2 = i$$

is

$$\exp(i3\pi/4) = \frac{1}{\sqrt{2}}(-1 - i).$$