5. Holomorphic functions defined by series

Definition 5.1. *Let $f\colon U \longrightarrow \mathbb{C}$ be a function defined on some open subset of $\mathbb{C}$. We say that f is* ***analytic at*** *$a \in U$ if there is a sequence of complex numbers $a_0, a_1, a_2, \dots$ such that*

$$f(z) = \sum_{n\in\mathbb{N}} a_n(z-a)^n$$

in some neighbourhood of p.

We say that f is ***analytic on*** *U if it is analytic at every point of U.*

Definition-Lemma 5.2. *Let $\sum a_n(z-a)^n$ be a power series, where $a_0, a_1, a_2, \dots$ are complex numbers. Then there is a real number $0 \leq R \leq \infty$, called the* ***radius of convergence*** *with the following properties:*

1. *For every $|z-a| < R$, the series converges absolutely.*
2. *For every $|z-a| > R$, the series diverges.*
3. *For every $\rho < R$, the series converges uniformly in the disc $|z-a| < \rho$.*

Further the number R satisfies

$$\frac{1}{R} = \limsup |a_n|^{1/n}.$$

Proof. We may suppose that $a = 0$. We first show (1) and (3). Suppose $\rho < R$ and $|z| < \rho$. Then, for n large enough,

$$|a_n|^{1/n} < 1/\rho.$$

Hence, for n large enough,

$$|a_n z^n| = (|a_n|^{1/n}|z|)^n < (|z|/\rho)^n.$$

But $|z|/\rho < 1$ and so the series $\sum_n |a_n z^n|$ is dominated by a uniformly convergent geometric series. Hence (1) and (3).

Suppose that $|z| > \rho > R$. Then, for infinitely many n,

$$|a_n|^{1/n} > (1/R)(R/\rho)$$

Hence, for infinitely many n,

$$|a_n z^n| = (|a_n|^{1/n}|z|)^n > ((1/R)(R/\rho)\rho)^n = 1.$$

But then (2) holds, as the terms of a convergent sum tend to zero. □

Consider the real power series

$$1 - x + x^2 - x^3 + \dots$$

The radius of convergence is 1, and this series diverges at ± 1. In fact this is a geometric series, whose sum is

$$\frac{1}{1+x}.$$

Thus it is not at all surprising that this series diverges for $x = -1$, since the corresponding function is not defined there. However if one replaces x by x^2, this will not change the radius of convergence, even though the function

$$\frac{1}{1+x^2}$$

is defined at $x = \pm 1$.

However if we replace x by z and work in the complex plane, then if we look at

$$\frac{1}{1+z^2}$$

we see that $z = \pm i$ are two points on the circle of convergence where the function is not defined.

Lemma 5.3.

$$\lim_{n\to\infty} (n+1)^{1/n} = 1.$$

Proof. Taking logs, it suffices to observe that

$$\lim_{n\to\infty} \frac{\log(n+1)}{n} = 0. \qquad \square$$

Proposition 5.4. *The analytic function $f(z) = \sum a_n(z-a)^n$ is holomorphic inside the region $|z-a| < R$. Furthermore the derivative is given by the power series*

$$f'(z) = \sum_n n a_n (z-a)^{n-1}$$

in the same region.

In particular every analytic function is infinitely differentiable.

Proof. As before we may as well set $a = 0$. Consider the series $\sum b_n z^n$, where $b_n = (n+1)a_{n+1}$. Then the radius of convergence of this series is equal to the inverse of the limit

$$\begin{aligned} \limsup |b_n|^{1/n} &= \limsup (n+1)^{1/n} |a_n| \\ &= \limsup (n+1)^{1/n} \limsup |a_n| \\ &= \frac{1}{R}, \end{aligned}$$

where we used (5.3). Thus the power series $\sum b_n z^n$ converges in the circle $|z| < R$ and we may define a function

$$g(z) = \sum_{n \in \mathbb{N}} b_n z^n.$$

Suppose that we set $s_n(z)$ equal to the first $n+1$ terms of the power series expansion for f and let $R_n(z)$ be the rest, so that

$$f(z) = s_n(z) + R_n(z).$$

Then $g(z) = \lim_{n\to\infty} s'_n(z)$. Consider

$$\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) = \left(\frac{s_n(z) - s_n(z_0)}{z - z_0} - s'_n(z_0)\right) + (s'_n(z_0) - g(z_0)) + \left(\frac{R_n(z) - R_n(z_0)}{z - z_0}\right)$$

for any $|z_0| < R$. Since $s'_n(z_0)$ converges to $g(z_0)$

$$s'_n(z_0) - g(z_0) < \epsilon/3$$

for all n sufficiently large. The last term is

$$\sum_{k=n}^{\infty} a_{k+1} \frac{z^{k+1} - z_0^{k+1}}{z - z_0} = \sum_{k=n}^{\infty} a_{k+1}(z^k + z^{k-1}z_0 + \dots + z z_0^{k-1} + z_0^k),$$

and so

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| \leq \sum_{k=n}^{\infty} (k+1)|a_{k+1}|\rho^k.$$

Pick $\epsilon > 0$. As the expression on the right is the tail of a convergent series, we may find n sufficiently large so that the expression is less than $\epsilon/3$. On the other hand we may find $\delta > 0$ such that the first term is less than $\epsilon/3$ for all $|z - z_0| < \delta$. Thus

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - g(z_0)\right| \leq \epsilon,$$

and we are done. □