## 3. Polynomials and Rational Functions

Note that the function $z$ is holomorphic. Since the sum, product and scalar multiple of any holomorphic functions is holomorphic, it follows that any polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

is holomorphic.

By the Fundamental Theorem of Algebra (to be proved later) we can completely factor $P(z)$,

$$P(z) = a_n (z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \cdots (z - \alpha_n).$$

Recall that if we write

$$P(z) = (z - \alpha)^k Q(z)$$

where $Q(z)$ is a polynomial and $Q(\alpha) \neq 0$, then we say that $\alpha$ is a **zero of order** $k$. By the standard rules of calculus the order $k$ is determined by the prescription

$$P(\alpha) = \cdots = P^{(k-1)}(\alpha) = 0$$

and $P^{(k)}(\alpha) \neq 0$.

**Proposition 3.1.** *The zeroes of $P'(z)$ lie in the smallest convex polygon determined by the zeroes of $P(z)$.*

*Proof.* It is a standard result in convex geometry that a closed convex set is the intersection of the half spaces that contain it. Therefore it suffices to prove that if all the zeroes of $P(z)$ lie in some half space $H$, then the zeroes of $P'(z)$ lie in the same half space. Translating and rotating we may as well assume that this half space is given by the imaginary part of $z$ is greater than zero.

Now

$$f(z) = \frac{P'(z)}{P(z)} = \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_2} + \cdots + \frac{1}{z - \alpha_n}.$$

Suppose that the imaginary part of $z$ is less than zero. Then a zero of $P'(z)$ is a zero of the rational function $f(z)$, as the denominator $P(z)$ is not zero.

On the other hand, by assumption every $\alpha_i$ has imaginary part greater than zero. Hence the imaginary part of every $z - \alpha_i$ is less than zero. Now the imaginary part of the reciprocal of a complex number has opposite sign and so every term in the sum above has positive imaginary part. In this case $f(z)$ cannot be zero, as its imaginary part is not zero. $\square$

Consider the rational function $f(z) = P(z)/Q(z)$, given as the quotient of two polynomial functions $P(z)$ and $Q(z)$. Cancelling common factors, we may as well assume that $P(z)$ and $Q(z)$ are never both zero. Note that a rational function $P(z)/Q(z)$ is holomorphic, wherever the denominator is not zero and we have the usual formula for the derivative. We call a zero $\alpha$ of $Q(z)$ a **pole** of $P(z)/Q(z)$. The **order of the pole** $\alpha$ is equal to the order of $\alpha$ as a zero of $Q(z)$.

To further understand rational functions, it is useful to work on the extended complex plane, $\mathbb{C} \cup \{\infty\}$. That is, we formally add a number $\infty$ and we set $f(\alpha) = \infty$, wherever $\alpha$ is a pole of $f$. The natural question then becomes, can we assign a value to $f(\infty)$? O ne approach might be to take a limit. The problem with this is that one cannot recover the order of $f$ at infinity (supposing for example that $f(\infty) = 0$). The correct approach is to use the coordinate $1/z$ at infinity.

For example, suppose

$$f(z) = \frac{z+1}{z^2-2}.$$

Consider $g(z) = f(1/z)$. We want to compute $g(0)$. Then

$$g(z) = f(1/z) = \frac{1/z+1}{(1/z)^2-2} = \frac{z(1+z)}{1-2z^2}.$$

So $f(\infty) = g(0) = 0$ and the order is 1.

There is another way to consider the extended complex plane. Suppose that you take the sphere $x^2 + y^2 + z^2 = 1$ inside $\mathbb{R}^3$. Take the point $p = (0, 0, 1)$ (the North pole). Consider projection from this point down to the plane $(x, y, 0)$ (that is, the plane $z = 0$).

This map is well-defined at any point of the sphere other than $p$.

Now a line can only meet a sphere in at most two points. Thus this map is injective. Now suppose that one takes a horizontal plane that lies somewhere between the North and South pole. This plane will cut the sphere in a circle. Consider the image of this circle. It is easy to see that we get a circle in the plane, with centre the origin.

When the plane is at the South pole, this circle has zero radius. When the plane approaches the North pole, a moments thought will convince the reader that the radius of this circle is approaching infinity. Clearly the function which assigns to the height of the plane above the South pole, the radius of the corresponding circle, is a continuous function. By the intermediate value Theorem, we therefore get all possible radii. Thus this map is also surjective.

Thus the points of $\mathbb{C} = \mathbb{R}^2$ are in bijection with the points of the sphere minus the origin. Mapping the north pole to the point at infinity, we get a bijection between the extended complex plane and the Riemann sphere.

The important point about this map is that it preserves angles. Thus if you take two curves on the Riemann sphere and look at their images in the plane, then the angle between them has not changed. A map that preserves angles is called **conformal**. We will see later that this is a defining property of holomorphic maps.

It is interesting to figure out images of various locii in the sphere. We have already seen that the South pole is sent to the origin and the North pole is sent to infinity. The equator is sent to the unit circle (this is clear, as the the plane $z = 0$ cuts the sphere in the unit circle). In particular $(1, 0, 0)$ is sent to 1 and $(0, 1, 0)$ is sent to $i$.

There is yet another way to look at this. The projective line $\mathbb{P}^1$ denotes the set of one dimensional linear subspaces of $\mathbb{C}^2$. Note that given any point of $\mathbb{C}^2 - \{0\}$, we get a line in $\mathbb{C}^2$ through the origin. On the other hand two points which give the same line are non-zero scalar multiples of each other and so

$$\mathbb{P}^1 = \frac{\mathbb{C}^2 - \{0\}}{\mathbb{C} - \{0\}}.$$

Thus a point of $\mathbb{P}^1$ is represented by an equivalence class $(u, v)$ of non-zero vectors. We denote this equivalence class by $[u : v]$. Suppose that $v \neq 0$. Then we may rescale $v$ so that it is equal to one. Thus $[u : v] = [u/v : 1]$. Now the ratio $u/v$ can take on any value in $\mathbb{C}$. Thus $\mathbb{P}^1$ contains a copy of $\mathbb{C}$. Suppose that $v = 0$. Then $u \neq 0$. Thus $[u : v] = [u : 0] = [1 : 0]$. Thus $\mathbb{P}^1$ is a copy of $\mathbb{C}$ with a point added.

Suppose that we introduce coordinates $X$ and $Y$ on $\mathbb{C}^2$ (note that $X$ and $Y$ are complex coordinates). We call $X$ and $Y$ homogeneous coordinates on $\mathbb{P}^1$. Note that $X$ and $Y$ are not coordinates on $\mathbb{P}^1$, since on $\mathbb{P}^1$ they are only well defined up to non-zero scalars. The only thing we can ask is if they are zero or not.

However the ratio $z = X/Y$ is an honest coordinate on $\mathbb{P}^1$, wherever $Y \neq 0$. In fact $z \colon \mathbb{P}^1 - [1 : 0] \longrightarrow \mathbb{C}$ is exactly the identification of $\mathbb{C}$ with the subset of $\mathbb{P}^1$ we obtained before.

On the other hand, the ratio $Y/X$ is an honest coordinate on $\mathbb{P}^1$, wherever $X \neq 0$, that is, on $\mathbb{P}^1 - [0 : 1]$. But

$$Y/X = 1/(X/Y) = 1/z.$$

Thus we have exhibited explicit bijections between the extended complex plane, the Riemann sphere and $\mathbb{P}^1$.