22. Representations of Meromorphic Functions

There are two natural ways to represent a rational function. One is to express it as a quotient of two polynomials, the other is to use partial fractions. The object is to do the same for meromorphic functions. As a first try, one might consider

$$f(z) = \sum_k P_k\left(\frac{1}{z-b_k}\right) + g(z),$$

where b_k are the poles, and P_k is a polynomial that is called the **singular part** of $f(z)$ at b_k and $g(z)$ is, of course, holomorphic. In general the problem is that the sum of the singular parts need not converge. Nevertheless, it turns out that the sum of the singular parts does often converge and in particular cases, g has a very nice representation.

In fact it is always the case that we can find polynomials p_k such that the series

$$\sum_k P_k\left(\frac{1}{z-b_k}\right) - p_k(z),$$

converges. We will prove this in the case the region is the whole complex plane.

We first need a result about the remainder in Talyor's Theorem:

Theorem 22.1. *If $f(z)$ is holomorphic in a region U, containing a, then we may write*

$$f(z) = f(a)+f'(a)(z-a)+\frac{f''(a)}{2}(z-a)^2+\dots+\frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1}+f_n(z)(z-a)^n,$$

where

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(w)\,\mathrm{d}w}{(w-a)^n(w-z)}$$

is holomorphic and C is any circle centred around a contained in U.

Proof. Everything is clear except for the statement about the expression for $f_n(z)$. Since $f_n(z)$ is holomorphic we have

$$f_n(z) = \frac{1}{2\pi i}\int_C \frac{f_n(w)\,\mathrm{d}w}{(w-z)},$$

by Cauchy's integral formula. If we take the expression for $f(z)$ above and replace z by w and solve for $f_n(w)$ we get

$$f_n(w) = \frac{f(w)}{(w-a)^n} - \frac{f(a)}{(w-a)^n} - \frac{f'(a)}{(w-a)^{n-1}} - \dots - \frac{f^{(n-1)}(a)}{(n-1)!(w-a)}.$$

If we multiply by the reciprocal of $w - z$ and integrate over C, then ignoring constant factors, every term but the first has the form

$$F_i(a) = \int_C \frac{\mathrm{d}w}{(w-a)^i(w-z)},$$

for $i \geq 1$. But

$$F_1(a) = \frac{1}{z-a}\int_C \frac{1}{w-z} - \frac{1}{w-a}\,\mathrm{d}w = 0$$

is holomorphic and so

$$i!F_{i+1}(a) = F_1^{(i)}(a) = 0. \qquad \square$$

Theorem 22.2. *Let b_k be a sequence of complex numbers which tends to infinity and let P_k be polynomials without constant term.*

Then there are meromorphic functions with the given singular part at b_k:

$$P_k\left(\frac{1}{z-b_k}\right).$$

Moreover, any such function has the form

$$f(z) = \sum_k \left[P_k\left(\frac{1}{z-b_k}\right) - p_k(z)\right] + g(z),$$

where p_k are polynomials and g is entire.

Proof. Assume that no b_k is zero. Let $p_k(z)$ be the Taylor polynomial of

$$P_k\left(\frac{1}{z-b_k}\right),$$

up to a certain degree n. Then (22.1) applied to the circle $|z| = \frac{|b_k|}{2}$ implies that

$$\left|P_k\left(\frac{1}{z-b_k}\right) - p_k(z)\right| \leq 2M\left(\frac{2|z|}{|b_k|}\right)^{n+1},$$

for all $|z| \leq |b_k|/4$, where M is the maximum of P_k in the disc $|z| \leq |b_k|/2$.

Now choose n sufficiently large so that the RHS is less than 2^{-k}. Thus the sum of the singular parts converges. Moreover, the series converges uniformly in the disc $|z| \leq R$, if we omit the terms with $|b_k| \leq 4R$. Thus by Weierstrass' Theorem (applied to concentric discs whose radius goes to infinity) g is an entire holomorphic function. $\square$

It is interesting to see how this works in practice. Consider the function

$$f(z) = \frac{\pi^2}{\sin^2 \pi z}.$$

Then $f(z)$ has double poles at the points $z = n$. Now at $z = 0$ the singular part is

$$\frac{1}{z^2}.$$

As $\sin^2 \pi(z - n) = \sin^2 \pi z$, it follows that the singular part at $z = n$ is

$$\frac{1}{(z-n)^2}.$$

Now the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

is convergent for $z \neq n$, simply by comparing this sequence with

$$\sum \frac{1}{n^2}.$$

Moreover it is uniformly convergent on any compact set which omits the singular parts on that set. Thus

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} + g(z),$$

where $g(z)$ is entire. I claim that $g(z)$ is zero.

Now both sides, possibly omitting $g(z)$, are periodic, with period one. Thus $g(z)$ is periodic, with period one. If we write $z = x + iy$ then

$$\sin \pi z = \frac{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{-\pi y}}{2i}$$

and so

$$\overline{\sin \pi z} = \frac{e^{-i\pi x}e^{-\pi y} - e^{i\pi x}e^{-\pi y}}{-2i}.$$

Taking the product we get four terms that can be paired to give

$$|\sin \pi z|^2 = \cosh^2 \pi y - \cos^2 \pi x.$$

where we added and subtracted $1/2$.

If we fix x then the LHS tends uniformly to zero as $|y| \to \infty$. Now the same holds for any term of the sum

$$\frac{1}{(z-n)^2}.$$

As the sum converges uniformly for $|y| \geq 1$, it follows that the infinite sum tends uniformly to zero as $|y| \to \infty$. Thus $g(z)$ tends uniformly to zero as $|y|$ tends to infinity for fixed x. Therefore $g(z)$ is bounded on the strip $0 \leq |x| \leq 1$. But then g is bounded, by periodicity and so $g(z)$ is a constant by Liouville. As it tends to zero, it must be zero. Thus

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}.$$

Suppose that we try to integrate both sides to get a new identity. The LHS is the derivative of $-\pi \cot \pi z$. The term

$$\frac{1}{(z-n)^2}$$

on the RHS is the derivative of

$$\frac{-1}{z-n}.$$

Unfortunately the sum

$$\sum \frac{1}{z-n},$$

diverges. If we subtract the first term of the Taylor series we get a convergent series, as

$$\sum_{n\neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \sum_{n \neq 0} \frac{z}{n(z-n)},$$

is comparable to $\sum 1/n^2$. This series converges uniformly on compact subsets, provided we omit all singular parts and thus, differentiating term by term, we get

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right),$$

up to a constant. Now clumping positive and negative terms together we get

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n>0} \left(\frac{2z}{z^2 - n^2} \right).$$

Now both sides are clearly odd functions. Thus the additive constant is in fact zero.

Let's try to go backwards. Suppose we start with

$$\lim_{m \to \infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n} = \frac{1}{z} + \sum_{n \neq 0} (-1)^n \frac{2z}{z^2-n^2}.$$

Then this represents a meromorphic function.

If we separate the odd and even terms we get:

$$\sum_{n=-2m-1}^{2m+1} \frac{(-1)^n}{z-n} = \sum_{n=-m}^{m} \frac{1}{z-2n} - \sum_{n=-m-1}^{m} \frac{1}{z-1-2n}.$$

Taking the limit we get

$$\frac{\pi}{2} \cot \frac{\pi z}{2} - \frac{\pi}{2} \cot \frac{\pi(z-1)}{2} = \frac{\pi}{\sin \pi z},$$

so that

$$\frac{\pi}{\sin \pi z} = \lim_{m\to\infty} \sum_{n=-m}^{m} \frac{(-1)^n}{z-n}.$$