2. Holomorphic and Harmonic Functions

Basic notation. Considering $\mathbb{C}$ as $\mathbb{R}^2$, with coordinates x and y, $z = x + iy$ denotes the standard complex coordinate, in the usual way.

Definition 2.1. *Let $f \colon U \longrightarrow \mathbb{C}$ be a complex valued function defined on some open subset $U \subset \mathbb{C}$.*

*We say that $f(z)$ is **differentiable** at z_0, with **derivative** $f'(z_0)$, if*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

*We say that f is **holomorphic** if it is differentiable at every point of U*

It is easy to prove, as in the real case, that the sum, difference, product, quotient, and composition of holomorphic functions is holomorphic and that the usual formulas apply.

The obvious first step in analyzing the condition that the derivative exists, is to see what happens if we approach z_0 along a line, in particular along a horizontal and vertical line.

Along a horizontal line, we fix $y = y_0$ and vary x. Using the obvious notation $z_0 = x_0 + iy_0$, $f(z) = f(x, y)$ and setting $z = x + iy_0$ we get

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \\ &= \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}. \end{aligned}$$

Similarly along a vertical line, we fix $x = x_0$ and vary y. Setting $z = x_0 + iy$, we get

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{i(y - y_0)} \\ &= \left.\frac{1}{i}\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.-i\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}. \end{aligned}$$

In other words, if f is holomorphic then

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \left.-i\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}.$$

There is another way to interpret all of this. Suppose we start with $f(x, y)$, a function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Now we can express x and y in terms of z and $\bar{z}$,

$$x = \frac{1}{2}(z + \bar{z}) \qquad \text{and} \qquad y = \frac{1}{2i}(z - \bar{z}) = -\frac{i}{2}(z - \bar{z}).$$

Thus we can consider $f(x, y)$ as a function $f(z, \bar{z})$ of z and $\bar{z}$. Now formally differentiating and formally applying the chain rule we would have

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y},$$

and

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}.$$

This suggests that we introduce operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

In these terms, the condition above becomes

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

In other words, a function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is holomorphic if and only if "it does not depend on $\bar{z}$".

Separating into real and imaginary parts, we get the famous Cauchy-Riemann equations. Suppose that we write $f = u + iv$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Lemma 2.2. *If*

$$u\colon U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

is a $\mathcal{C}^1$-function then

$$\epsilon = u(x + h, y + k) - \frac{\partial u}{\partial x}h - \frac{\partial u}{\partial y}k$$

goes to zero faster than (h, k) goes to zero, for any $(x, y) \in U$.

Proof. By assumption

$$\delta = u(x + h, y) - u(x, y) - \left.\frac{\partial u}{\partial x}\right|_{(x,y).} h.$$

goes to zero faster than h. On the other hand,

$$\eta = u(x + h, y + k) - u(x + h, y) \left.\frac{\partial u}{\partial y}\right|_{(x+h,y)} k.$$

goes to zero faster than k. As the partials are continuous, the difference

$$\left.\frac{\partial u}{\partial y}\right|_{(x+h,y)} - \left.\frac{\partial u}{\partial y}\right|_{(x,y)}$$

goes to zero as h goes to zero. So

$$\zeta = \left(\left.\frac{\partial u}{\partial y}\right|_{(x+h,y)} - \left.\frac{\partial u}{\partial y}\right|_{(x,y)} \right) k$$

goes to zero faster than k. Thus

$$\begin{aligned}
\epsilon &= u(x+h, y+k) - u(x,y) - \frac{\partial u}{\partial x}h - \frac{\partial u}{\partial y}k \\
&= u(x+h, y+k) - u(x+h, y) - \frac{\partial u}{\partial y}k + u(x+h, y) - u(x,y) - \frac{\partial u}{\partial x}h \\
&= u(x+h, y+k) - u(x+h, y) - \left.\frac{\partial u}{\partial y}\right|_{(x+h,y)} + \zeta + \delta \\
&= \eta + \zeta + \delta,
\end{aligned}$$

which goes to zero faster than (h, k). □

Proposition 2.3. *A function $f(z)$ is holomorphic if and only if its real and imaginary parts $u(x, y)$ and $v(x, y)$ are $\mathcal{C}^1$ (that is, the first derivatives exist and are continuous) and satisfy the Cauchy-Riemann equations.*

Proof. We have already seen that the derivatives of u and v exist and satisfy the Cauchy-Riemann equations. We will see later that if f is holomorphic then the derivative of f is holomorphic (so that f is infinitely differentiatible) and this will imply that u and v are $\mathcal{C}^1$.

Suppose that u and v are $\mathcal{C}^1$.

(2.2) implies that

$$\begin{aligned}
u(x+h, y+k) - u(x,y) &= \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \epsilon_1 \\
v(x+h, y+k) - v(x,y) &= \frac{\partial v}{\partial x}h + \frac{\partial v}{\partial y}k + \epsilon_2
\end{aligned}$$

where ϵ_1 and ϵ_2 go to zero faster than $h + ik$.

$$\begin{aligned}
f(x+h, y+k) - f(x,y) &= (u(x+h, y+k) - u(x,y)) + i(v(x+h, y+k) - v(x,y)) \\
&= \frac{\partial u}{\partial x}h + \frac{\partial u}{\partial y}k + \epsilon_1 + i\frac{\partial v}{\partial x}h + i\frac{\partial v}{\partial y}k + \epsilon_2 \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)h + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)k + \epsilon_1 + \epsilon_2 \\
&= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(h + ik) + \epsilon_1 + \epsilon_2
\end{aligned}$$

As $\epsilon_1 + \epsilon_2$ approaches zero faster than $h + ik$, it follows that f is differentiable. $\square$

It is interesting to further investigate the properties satisfied by u and v.

Recall that the **Laplacian** ∇ of a real function $u(x, y)\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined to be

$$\nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},$$

and that function u is said to be **Harmonic** if it is $\mathcal{C}^2$ and the Laplacian vanishes, that is

$$\nabla u = 0.$$

Now suppose that $f = u + iv$, where f is holomorphic, so that u is the real part of a holomorphic function. Latter we will show that f is infinitely differentiable, so that in particular u and v are both $\mathcal{C}^2$. On the other hand

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 v}{\partial x \partial y} \\
&= \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2},
\end{aligned}$$

where we used the fact that if v is $\mathcal{C}^2$ then its mixed partials are equal.

Thus

$$\nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and so u is harmonic. Similarly v is harmonic. Two functions u and v that satisfy the Cauchy-Riemann equations, are called **conjugate harmonic** and v is said to be the conjugate harmonic function of u. Note that this is a slight abuse of language, since v is only determined up to a constant.

Given a harmonic function u it is instructive to try to construct the harmonic conjugate v.

For example $u = x^2 - y^2$ is harmonic. $\frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = -2y$. Thus the harmonic conjugate satisfies

$$\frac{\partial v}{\partial x} = 2y \qquad \text{and} \qquad \frac{\partial v}{\partial y} = 2x.$$

Take the first equation and integrate with respect to x. We get that $v(x, y) = 2xy + g(y)$ where g is an arbitrary function of y. Now differentiate with respect to y. Then

$$2x + g'(y) = 2x.$$

Thus $g(y)$ is constant. Thus the harmonic conjugate of u is $v = 2xy$. Of course in this case the corresponding holomorphic function is

$$f(z) = u(x, y) + iv(x, y) = (x^2 - y^2) + i2xy = (x + iy)^2 = z^2.$$

In fact there is another way to compute harmonic conjugates, which does not depend on using integration. In other words, we will find a formula for a holomorphic function $f(z)$ in terms of its real part $u(x, y)$.

The trick is to consider the function $g(z, \bar{z}) = \overline{f(z)}$. Note that

$$\frac{\partial g}{\partial z} = \overline{\frac{\partial \bar{g}}{\partial \bar{z}}} = \overline{\frac{\partial f}{\partial \bar{z}}} = 0,$$

so that $g(z, \bar{z}) = g(\bar{z})$ is a function only of $\bar{z}$ (g is sometimes known as an antiholomorphic function).

Then

$$u(x, y) = \frac{1}{2}\left(f(x + iy) + g(x - iy)\right).$$

Now treat this equality formally and extend it to the case where $x = z/2$, $y = z/2i$. Thus

$$u(z/2, z/2i) = \frac{1}{2}\left(f(z) + g(0)\right).$$

Now $f(z)$ is determined up to an imaginary constant. Thus we may as well assume that $f(0)$ is real. In this case, we may take $f(0) = g(0) = u(0, 0)$. Thus

$$f(z) = 2u(z/2, z/2i) - u(0, 0).$$

For example, suppose that $u(x, y) = x^2 - y^2$. Then

$$f(z) = 2u(z/2, z/2i) - u(0, 0) = 2(z/2)^2 - 2(z/2i)^2 = z^2/2 + z^2/2 = z^2.$$

Note that for this to work, the expression $u(z/2, z/2i)$ has to make sense (for example, u is given as a power series).