14. Winding Numbers

We want to generalise Cauchy's Integral Formula to a more general path. If one looks at the proof of Cauchy's Integral Formula, the key point is to determine the integral

$$\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$

where γ is a closed path that does not contain a.

Definition-Lemma 14.1. *Let γ be a closed path and let a be a point off γ.*

*The **winding number** $n(\gamma; a)$ of γ about a is equal to the integer*

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z.$$

Proof. The only thing to prove is that we do get an integer. One way to proceed is to recognize $\frac{1}{z-a}$ as the derivative of $\log(z-a)$. The result is then clear, except for the fact that we need to be careful about the definition of the argument, since the whole point is that we will wrap around a. This argument can be made precise, but it is a little delicate.

A more direct way is to proceed by direct computation. Let $\gamma(t)$ be a parametrisation of the path. Then the integral is

$$\frac{1}{2\pi i}\int_c^d \frac{\gamma'(t)}{\gamma(t)-a}\,\mathrm{d}t.$$

Define a function

$$g(t) = \int_c^t \frac{\gamma'(s)}{\gamma(s)-a}\,\mathrm{d}s.$$

Then

$$g'(t) = \frac{\gamma'(t)}{\gamma(t)-a}.$$

Thus

$$\begin{aligned}\frac{d}{dt}(e^{-g}(\gamma-a)) &= e^{-g}(\gamma') - g'e^{-g}(\gamma-a)\\ &= e^{-g}\left(\gamma' - g'(\gamma-a)\right)\\ &= e^{-g}(\gamma'-\gamma')\\ &= 0.\end{aligned}$$

Thus $e^{-g}(\gamma-a)$ is constant. At $t=c$ we get $\gamma(c)-a$ and at $t=c$ we get

$$e^{-g(d)}(\gamma(d)-a) = e^{-g(d)}(\gamma(c)-a).$$

Thus $e^{-g(d)}=1$ and so $g(d)=2\pi i k$ for some integer k. □

The winding number satisfies some obvious basic properties. For example suppose that γ and γ' are two closed paths, then

$$n(\gamma + \gamma'; a) = n(\gamma; a) + n(\gamma'; a).$$

Lemma 14.2. *Let γ be a closed path.*

Then the winding number is constant on the components of $\mathbb{C} - \gamma$ and it is zero on the unbounded component.

Proof. To prove the first statement it suffices to prove that the winding number is a continuous function of the second argument.

$$\begin{aligned}
n(\gamma; a) - n(\gamma; b) &= \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a} \, \mathrm{d}z - \frac{1}{2\pi i} \int_\gamma \frac{1}{z-b} \, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_\gamma \frac{a-b}{(z-a)(z-b)} \, \mathrm{d}z \\
&= \frac{a-b}{2\pi i} \int_\gamma \frac{1}{(z-a)(z-b)} \, \mathrm{d}z \\
&\leq |a-b| \frac{1}{2\pi} L(\gamma) M.
\end{aligned}$$

where M is the maximum value of $\frac{1}{|z-a||z-b|}$ on the closed path γ. As the image of γ is compact, for any a not on γ there is a small ball containing a such that M is bounded, for any a and b belonging to this ball. It is then clear that given any ϵ we may choose the difference as small as we please.

On the other hand the value of the integral $n(\gamma; a)$ may be made as small as one pleases, if $|a|$ is sufficiently large. It follows that the integral is in fact zero, for $|a|$ sufficiently large and so the winding number is zero on the unbounded component. □

Example 14.3. *Let γ be a circle and let a be a point off the circle. Then*

$$n(\gamma; a) = \begin{cases} 1 & \textit{if } a \textit{ is in the interior of the circle} \\ 0 & \textit{otherwise.} \end{cases}$$

Indeed, $\mathbb{C} - \gamma$ has two components. To compute what happens on the bounded component, we may choose a to be the centre of the circle, when the result has already been proved. Otherwise a belongs to the unbounded component and the integral is zero.

Theorem 14.4 (Cauchy's Integral Formula: bis)**.** *Let γ be a circle which contains a in its interior and let $f(z)$ be a holomorphic function on the circle.*

Then

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, \mathrm{d}z$$

Proof. Consider the function

$$g(z) = \frac{f(z) - f(a)}{z-a}.$$

Then $g(z)$ satisfies the hypothesis of (13.5). Thus

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \frac{f(z)}{z-a}\, dz - \int_\gamma \frac{f(a)}{z-a}\, dz.$$

But by (14.3) the latter integral is $2\pi i f(a)$. □