12. Complex Integration

First some basic definitions:

Definition 12.1. *Let X be a topological space. A* **curve** *in X is a continuous function $\gamma\colon [a,b] \longrightarrow X$.*

There are obvious notions attached to this definition; the image (aka trace), endpoints, writing a curve as a sum (aka composing paths),

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_k,$$

the reversed curve, parametrisation, all of which are left to the reader to devise.

Definition 12.2. *Let U be a region inside $\mathbb{C}$. A curve γ is said to be **piecewise differentiable** if there are finitely many differentiable curves γ_i, such that γ is the sum of the γ_i.*

Definition 12.3. *Let γ be a piecewise differentiable curve and let f be a continuous function, defined on a region U, containing the image of γ. The integral of γ, written*

$$\int f \,\mathrm{d}\gamma,$$

is defined as

$$\int_a^b f\left(\gamma(t)\right)\gamma'(t)\,\mathrm{d}t.$$

We use other obvious notation for this integral. For example,

$$\int_\gamma f(z)\,\mathrm{d}z.$$

The integral is linear, both in f and in γ. Note that the integral is invariant under reparametrisation of γ.

Definition 12.4. *Let $\gamma\colon [a,b] \longrightarrow U$ be a piecewise differentiable curve. The **length** $L(\gamma)$ of γ is defined to be*

$$\int_a^b |\gamma'(t)|\,\mathrm{d}t.$$

Lemma 12.5. *Let γ be a piecewise differentiable curve and let f be a continuous function, defined on a region U, containing the image of γ. Let $M = \max|f(z)|$, where the maximum is taken over the image of γ.*
Then

$$\left|\int f\,\mathrm{d}\gamma\right| \leq ML(\gamma).$$

Proof. Clear, as

$$\left|\int f\,\mathrm{d}\gamma\right| \leq \int |f(t)|\cdot|\gamma'(t)|\,\mathrm{d}t \leq \int M|\gamma'(t)|\mathrm{d}\gamma = ML(\gamma). \qquad \square$$

Example 12.6. *Let z_0 and z_1 be two points of $\mathbb{C}$. The line segment z_0z_1 is the curve $\gamma\colon [0,1] \longrightarrow \mathbb{C}$ given as*

$$\gamma(t) = z_0 + t(z_1 - z_0).$$

Then the length of γ is $|z_1 - z_0|$.

Let $z_0 \in \mathbb{C}$. The circle with centre z_0 and radius ρ is the curve $\gamma\colon [0,1] \longrightarrow \mathbb{C}$ given as

$$\gamma(t) = z_0 + \rho e^{2\pi i t}.$$

Then the length of γ is $2\pi\rho$.

Finally suppose we have a rectangle, $a \leq \operatorname{Re} z \leq b$, $c \leq \operatorname{Im} z \leq d$. Parametrise the boundary into four straight line segments, starting at the bottom left and going around the boundary anti-clockwise. Clearly the length is $2(b-a) + 2(d-c)$.

Proposition 12.7. *Let f be a $\mathcal{C}^1$-function defined on a region U that contains a rectangle R.*

Then

$$\iint_R \frac{\partial f}{\partial \bar{z}}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2i}\int_{\partial R} f\,\mathrm{d}z.$$

Proof. We first replace the integrand by

$$\frac{\partial f}{\partial y} \qquad \text{and} \qquad \frac{\partial f}{\partial x}$$

and calculate each part separately:

$$\begin{aligned}
\iint_R \frac{\partial f}{\partial y}\,\mathrm{d}x\,\mathrm{d}y &= \int_a^b \mathrm{d}x \int_c^d \frac{\partial f}{\partial y}\,\mathrm{d}y \\
&= \int_a^b f(x,d) - f(x,c)\,\mathrm{d}x \\
&= -\left(\int_{\gamma_1+\gamma_3} f\,\mathrm{d}z\right).
\end{aligned}$$

Similarly

$$\begin{aligned}\iint_R \frac{\partial f}{\partial x}\,\mathrm{d}x\,\mathrm{d}y &= \int_c^d \mathrm{d}y \int_a^b \frac{\partial f}{\partial x}\,\mathrm{d}x \\ &= \int_c^d f(b,y) - f(a,y)\,\mathrm{d}y \\ &= \frac{1}{i}\left(\int_{\gamma_2+\gamma_4} f\,\mathrm{d}z\right).\end{aligned}$$

To get from the second line to the third line we used the fact that

$\mathrm{d}z = i\mathrm{d}y$ since on γ_2 (respectively γ_4) $z = b + iy$ ($z = d + iy$).

As

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

the result follows easily. □

Theorem 12.8. *(Cauchy's Formula) Let f be a holomorphic function defined on a region U that contains a rectangle R.*

Then

$$\int_{\partial R} f(z)\,dz = 0.$$

Proof. Note that we cannot apply (12.7), since we don't know that f is $\mathcal{C}^1$.

It is enough to prove this result in the case when the ratio of the sides of the rectangle is a rational number, since every rectangle is a limit of its interior rectangles.

First we divide the rectangle up into a grid of small squares, $R_1, R_2, \dots, R_m$. Then

$$\int_{\partial R} f(z)\,dz = \sum_i \int_{\partial R_i} f(z)\,dz.$$

In fact for each interior side, there are two squares that border this side, and the integrals over those squares will traverse that side in opposite directions. In this way the integrals cancel and we are left with the integral over the boundary.

As f is differentiable, for every point z_0 and for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \epsilon$$

for all

$$|z - z_0| < \delta.$$

Suppose that we could find a subdivision such that for every square of the subdivision, there is a point z_0 *belonging* to the square, such that the open ball of radius δ completely contains the square. Then for every square of the subdivision, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \phi(z)$$

where

$$|\phi(z)| \leq \epsilon |z - z_0|.$$

Now consider integrating this expression over a square. Since any function of the form $az + b$ is most certainly $\mathcal{C}^1$, by (12.7), the only term that contributes is the $\phi(z)$ term. If the side of the square is l then by (12.5) this term contributes at most

$$\epsilon 4\sqrt{2} l^2.$$

The sum of the integrals over all the squares would then be at most

$$4\sqrt{2}\epsilon(b - a)(d - c).$$

Therefore it suffices to check that we can find such a subdivision. Suppose not. Suppose that we are given a subdivision. Then we further subdivide as follows. We leave unchanged every square such that the given inequality holds in the square. Otherwise we further subdivide any square (say into four pieces).

Continuing in this way, we would find a sequence of nested squares such that there is no point belonging to each such square for which the inequality above holds. Since the squares are nested, we may pick a point z_0 in the intersection. For this point, there is a δ such that the inequality above holds.

On the other hand, z_0 belongs to each of these squares. As the lengths of these squares is going to zero, one of these squares is entirely contained in the open ball of radius δ, a contradiction. □

Lemma 12.9. *If γ is a square with centre a then*

$$\int_\gamma \frac{1}{|z - a|} \, \mathrm{d}z \leq 8.$$

Proof. We may assume that $a = 0$ in which case the vertices of the square are $(\pm l, \pm l)$. Let γ_1 be the side from $(-l, -l)$ to $(l, -l)$. Then

$$\begin{aligned}
\int_{\gamma} \frac{1}{|z-a|}\,\mathrm{d}z &= 4\int_{\gamma_1} \frac{1}{|z-a|}\,\mathrm{d}z \\
&= 4\int_{-l}^{l} \frac{1}{\sqrt{x^2+l^2}}\,\mathrm{d}x \\
&\leq 4\int_{-l}^{l} \frac{1}{\sqrt{l^2}}\,\mathrm{d}x \\
&= 8.
\end{aligned}$$

□

Proposition 12.10. *Let U be a region that contains a rectangle R. Let $f(z)$ be a function that is holomorphic outside finitely many points $a_1, a_2, \dots, a_k$ not on the boundary of the rectangle. In addition suppose that*

$$\lim_{z \to a_i} (z - a_i) f(z) = 0$$

for every i.

Then

$$\int_{\partial R} f(z)\,\mathrm{d}z = 0.$$

Proof. Breaking the rectangle up into smaller rectangles, we may suppose that $k = 1$. Set $a = a_1$. Subdividing the rectangle into nine rectangles, in such a way that there is one small square that contains a, and applying Cauchy's Theorem to the rectangles that don't contain a, we may assume that R is a square whose sides are arbitrarily small.

Thus given $\epsilon > 0$, we may assume that

$$|f(z)| \leq \frac{\epsilon}{|z-a|}.$$

On the other hand by (12.9)

$$\int_{\partial R} \frac{1}{|z-a|}\,\mathrm{d}z \leq 8.$$

As ϵ is arbitrary, the result follows. □