11. Level curves

Given a holomorphic function, one way to get a picture of what the function looks like, is to consider what happens to level curves, i.e. horizontal or vertical lines. If $f(z) = u(x, y) + iv(x, y)$, then we think of fixing x and varying y and vice-versa, both ways (vary x fix y, or fix u and vary v). Note that by conformality, level curves from different families are orthogonal to each other, that is they form an orthgonal net.

First let's go back to the function $f(z) = z^2$. If $z = x + iy$, then

$$u = x^2 - y^2,$$

and

$$v = 2xy.$$

Thus the level curves are

$$x^2 - y^2 = \mathrm{a}$$

and

$$2xy = \mathrm{b},$$

for suitable constants a and b.

Thus we get two families of orthogonal hyperbolas. These intersect at right angles, unless $a = b = 0$, in which case they intersect at an angle of $\pi/4$. But the derivative is zero there, so this does not contradict conformality of analytic functions.

Now consider

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Then

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right).$$

Thus the derivative is zero at ± 1. Note also that $f(0) = \infty$. Put $z = re^{i\theta}$ and $w = u + iv$. Then

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\theta,$$

and

$$v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\theta.$$

Eliminating θ, we get

$$\frac{u^2}{\frac{1}{4}\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r - \frac{1}{r}\right)^2} = 1.$$

Thus circles centred at the origin are mapped to ellipses. In fact this ellipse corresponds to the two circles

$$|z| = r$$

and

$$|z| = \frac{1}{r}.$$

Thus we get a double cover. In fact the axes tend to infinity as $r \to 0$ or $r \to \infty$. Thus in fact both the inside and the outside of the unit circle cover the plane. The unit circle $|z| = 1$ corresponds to the interval $[1, -1]$, described twice.

Thus the inverse function is not well-defined, unless one replaces the plane, by the Riemann surface, obtained by taking two copies of the plane, joined along a slit from $[-1, 1]$.