## 10. Comparing regions

**Definition 10.1.** *Let $U$ and $V$ be two regions. We say that*

$$f\colon U \longrightarrow V,$$

*is* ***biholomorphic*** *if $f$ is a bijection and both $f$ and its inverse are holomorphic.*

Note that $f$ is biholomorphic if and only if $f$ is a bijection and the derivative is nowhere vanishing.

For which regions $U$ and $V$ can we find $f$ biholomorphic? It is natural to start with the case when

$$V = \Delta = \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$$

is the interior of the unit disc.

Let's start with the case when

$$U = \{\, z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \,\}$$

is the upper half plane. In this case, let's try to find a Möbius Transformation carrying the upper half plane to the unit disc.

Now the $x$-axis will map to the unit circle. Thus two complex conjugate points $z$ and $\bar{z}$ will map to two inverse points, $w$ and $1/w$. In particular the origin and the point at infinity correspond to two complex conjugate points.

Suppose we try

$$w = \frac{az+b}{cz+d}.$$

Then $c \neq 0$, otherwise $\infty$ is fixed. In this case, $w = \infty$ corresponds to $-d/c$ and $w = 0$ corresponds to $-b/a$. Thus if

$$\alpha = -\frac{b}{a}$$

then

$$\bar{\alpha} = -\frac{d}{c}.$$

Thus

$$w = \frac{a}{c}\frac{z-\alpha}{z-\bar{\alpha}}.$$

Since the $x$-axis maps to the unit circle, the point $z = 0$ must correspond to a point of modulus one. Thus

$$\left|\frac{a}{c}\right| = 1.$$

So set

$$a = ce^{i\lambda}.$$

Then

$$w = e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}}.$$

Now $w = 0$ corresponds to $z = \alpha$. Thus $\alpha$ must lie in the upper half plane. Note that there are three real degrees of freedom. The real number $\lambda$, and the real and imaginary part of $\alpha$. In fact, with a little bit of work, one can show that one can choose three points of the real axis and map them to any three points of the unit circle.

Let's build on this example by considering some other functions. For example, suppose that we look at $f(z) = z^2$. Now if $z = re^{i\theta}$ then $w = \rho e^{i\phi} = r^2 e^{i2\theta}$. Thus the distance to the origin is squared and the angle is doubled. In particular the region

$$\alpha < \theta < \beta$$

is mapped to the region

$$2\alpha < \phi < 2\beta.$$

Note that if $\beta - \alpha > \pi$ then the new region intersects itself. Replacing the image, by two copies of the plane, joined along a slit from the origin, we remove this problem. In fact it was considerations of this type that originally lead Riemann to his study of Riemann surfaces.

Also circles centred at the origin, map to circles centred at the origin and half lines map to half lines. There is a similar story for $z \longrightarrow z^a$, where $a$ is real.

Finally consider the function

$$f(z) = e^z.$$

This maps horizontal lines to lines through the origin, and vertical lines to circles centred at the origin. Note that a horizontal strip,

$$a < y < b$$

where $b - a < 2\pi$ is mapped to the angular sector

$$a < \phi < b.$$

Now using this result and the fact that $e^{kz}$ maps vertical strips into angular regions, it follows that we may map any angular region into the unit circle, simply composing the logarithm with one of the transformations above.