1. Compare and contrast

The theory of functions of one real variable holds many nasty surprises, which in contrast to the theory of functions of one complex variable, might even be considered pathological.

For example it is easy to write down a function with as many derivatives as you please, but which is not infinitely differentiable. Indeed suppose you start with the standard step function

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0. \end{cases}$$

Then f is not continuous. Now let f_1 be the integral (say from $-\infty$ to x) of f and let g be the integral of f_1. Then f_1 is not differentiable at 0. But by the Fundamental Theorem of Calculus, g is differentiable, with derivative f_1. Integrating f as many times as you please, gives a function h with as many derivatives as you please.

A more subtle example is given by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

The derivative of f exists when $x \neq 0$ but it starts to oscillate wildly as x approaches zero, and it is easy to check that the derivative exists at the origin and is equal to zero. Thus f is differentiable, but its derivative is not continuous. Integrating as before, gives examples with as many derivatives as you please.

On the other hand, if f is a complex valued function, then if f has one derivative, then in fact it has infinitely many and they are all continuous.

At the opposite extreme, suppose that you start with the function

$$f(x) = \begin{cases} \exp(-1/x^2) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Then f is infinitely differentiable. Unfortunately every derivative at the origin is zero. So f does have a Taylor series, but the Taylor series is equal to zero, so that in fact f is not equal to its Taylor series.

On the other hand every holomorphic function (that is, a function of one complex variable which is differentiable) is analytic (that is, the function is equal to a power series).

Knowledge of a real function on an open subset, in no way determines its behaviour anywhere else. By contrast a holomorphic function is completely determined by its values on any infinite set.

There are plenty of $\mathcal{C}^\infty$ functions that are bounded, for example the standard trigonometric functions $\cos(x)$ and $\sin(x)$ lie between -1 and 1. By contrast any bounded holomorphic function is constant. In fact entire holomorphic functions can omit at most one value, and if they do omit this value, then they "spend a lot of time close to this value".

From a more sophisticated point of view, holomorphic functions display some remarkable features. For example, you can find a metric, such that under that metric, all holomorphic functions are distance decreasing.

Even from the perspective of trying to solve polynomial equations, functions of one complex variable are much better behaved than real functions. For example there are plenty of real polynomials with no real roots. The most one can say, is that every odd degree real polynomial equation has one real root. The fundamental Theorem of algebra states that every polynomial with complex roots can be completely factored over $\mathbb{C}$.

The reason for the good behaviour arises from the fact that it turns out to be an extremely strong condition that a complex function is differentiable.