HOMEWORK #1, DUE WEDNESDAY OCTOBER 9TH

1. Show that the function

$$f(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is infinitely differentiable and that $f^{(k)}(0) = 0$ for every k. Thus f is not analytic.

2. Show that the function

$$g(x) = \begin{cases} \exp(-\frac{1}{x^2}) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

is infinitely differentiable.

3. Consider the function

$$f(z) = \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Show that the real and imaginary parts satisfy the Cauchy-Riemann equations at $z = 0$, but that f is not analytic. (*Hint:* consider what happens as z approaches 0 along any line. Now consider what happens along an appropriate family of conics). Explain why this does not contradict the proposition proved in class.

4. If $f(z)$ and $g(z)$ are holomorphic, then prove that $f(g(z))$ is holomorphic.

5. For which values of a, b, c and d is the function $ax^3+bx^2y+cxy^2+dy^3$ harmonic? Find the harmonic conjugate in this case.