FINAL EXAM
MATH 220A, UCSD, AUTUMN 14

You have three hours.

There are 6 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:____

Signature:____

Problem	Points	Score
1	15	
2	15	
3	15	
4	30	
5	15	
6	10	
7	10	
8	10	
9	10	
Total	100	

1. (15pts) Prove that any bounded entire function $f(z)$ is constant.

Solution:
Let $f(z)$ be an entire function. Then there is a power series expansion of $f(z)$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Now suppose that $|f(z)| \leq M$. Then by Cauchy's Integral formula

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} \, \mathrm{d}z,$$

where γ is a circle of radius r centred at the origin. By assumption $|f(z)| \leq M$ on γ and so

$$\left| \frac{f(z)}{z^{n+1}} \right| \leq \frac{M}{r^{n+1}}.$$

It follows that

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}} \, \mathrm{d}z \right| \\ &\leq \frac{M}{r^{n+1}} r \\ &= \frac{M}{r^n}. \end{aligned}$$

Now let $r \to \infty$ to conclude that $a_n = 0$, for $n > 0$. But then $f(z)$ is constant.

2. (15pts) Give the definition of the residue of a function and state and prove the residue Theorem.

Solution:
Let $f(z)$ be a function, holomorphic in a punctured neighbourhood of a, with an isolated singularity at a. The residue $R = \operatorname{Res}_a f(z)$ of $f(z)$ at a is the unique number such that

$$f(z) - \frac{R}{z-a},$$

is the derivative of a holomorphic function, in a sufficiently small punctured neighbourhood of a.
Let $f(z)$ be a function, holomorphic on a region U, except for the points points $a_1, a_2, \ldots$, where f has isolated singularities. Let γ be a closed curve which has zero winding number about any point of $\mathbb{C} - U$ and which does not contain any of the points $a_1, a_2, \ldots$.
Then

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_i n(\gamma, a_i) \operatorname{Res}_{z=a_i} f(z).$$

As $a_1, a_2, \ldots$ has no accumulation point in U, we may assume that γ has non-zero winding number about only finitely many of the points $a_1, a_2, \ldots$. Pick small circles $\gamma_i \subset U$ about each such point. Let $\gamma' = \gamma - \sum n(\gamma, a_i)\gamma_i$. As each γ_i has zero winding number about each point of $\mathbb{C} - U$ it follows that γ has zero winding number about any point of $\mathbb{C} - (U - \{a_1, a_2, \ldots, a_k\})$. By Cauchy's Theorem, we have

$$\int_{\gamma'} f(z)\,\mathrm{d}z = 0.$$

On the other hand, by definition of the residue

$$\begin{aligned}\int_{\gamma_i} f(z)\,\mathrm{d}z &= R\int_{\gamma_i} \frac{1}{z-a}\,\mathrm{d}z \\ &= 2\pi i(\operatorname{Res}_{z=a_i} f(z)).\end{aligned}$$

The result follows easily.

3. (15pts) Let $f(z)$ be a function, meromorphic on a region U and let γ be a closed path that does not contain any of the zeroes or poles of $f(z)$. Suppose that the winding number of γ about every point is either zero or one and that the winding number of γ about any point of $\mathbb{C} - U$ is zero. Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, \mathrm{d}z = N - P,$$

where N is the number of zeroes and P is the number of poles, counted according to multiplicity.

Solution:
We try to apply the residue Theorem. It suffices to compute the residue at a zero or pole and show that this simply counts the order of the zero or pole, up to sign. Suppose that $f(z) = (z - a)^h g(z)$, where $g(z)$ is holomorphic and does not vanish at $z = a$. Then

$$\frac{f'(z)}{f(z)} = \frac{h}{z - a} + \frac{g'(z)}{g(z)}.$$

Thus the residue is by definition h and the result follows easily.

4. (30pts) Evaluate the following integrals.
(i)

$$\int_0^\infty \frac{x^2\,\mathrm{d}x}{x^4+5x^2+6}.$$

Solution:
We apply the residue Theorem to

$$f(z)=\frac{z^2}{z^4+5z^2+6}$$

around the closed path γ, which goes from 0 to R, around a semi-circle in the upper half plane to $-R$ and then from $-R$ to 0. As $f(x)$ is even the integral along the x-axis is twice the integral we want. If $|z|=R$, then

$$|f(z)|\leq \frac{R^2}{R^4-5R^2-6}$$

so that the integral along the semi-circle tends to zero, as R tends to infinity. Now

$$(z^4+5z^2+6)=(z^2+2)(z^3+3)=(z+\sqrt{2}i)(z-\sqrt{2}i)(z+\sqrt{3}i)(z-\sqrt{3}i).$$

Thus $f(z)$ has two poles in the upper half plane, one at $\sqrt{2}i$ and the other at $\sqrt{3}i$, both simple and the winding number of γ about these points is one. The residue at $\sqrt{2}i$ is

$$\lim_{z\to\sqrt{2}i} f(z)=\frac{-2}{(2\sqrt{2}i)1}=\frac{i\sqrt{2}}{2}.$$

Similarly the residue at $\sqrt{3}i$ is

$$\lim_{z\to\sqrt{3}i} f(z)=\frac{-3}{(2\sqrt{2}i)-1}=-\frac{i\sqrt{3}}{2}.$$

By the residue Theorem

$$\begin{aligned}\int_0^\infty \frac{x^2\,\mathrm{d}x}{x^4+5x^2+6}&=\frac{1}{2}\int_\gamma f(z)\,\mathrm{d}z\\&=\pi/2(\sqrt{3}-\sqrt{2}).\end{aligned}$$

(ii)

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,\mathrm{d}x.$$

Solution: Let

$$f(z) = \frac{z^{1/3}}{1+z^2}$$

and let γ be the closed path from ρ to R, along the x-axis, from R to $-R$ along a semi-circle in the upper half plane, from $-R$ to $-\rho$ along the x-axis and from $-\rho$ to ρ, along a semi-circle in the upper half plane, where $0 < \rho < R$. Here we choose the branch of $z^{1/3}$ using the branch of the logarithm, where the argument lies between $-\pi/2$ and $3\pi/2$.
When $|z| = R$,

$$|f(z)| \leq \frac{R^{1/3}}{R^2-1},$$

so that the integral along the large semi-circle tends to zero.
As for the integral around the small semi-circle, note that the argument is bounded and the length of the circle is going to zero. Thus the imaginary part of the integral is tending to zero. On the other hand, a similar analysis to the one above guarantees that the integral around the small semi-circle tends to zero as ρ tends to zero.
Finally note that the real parts of the integral from ρ to R and from $-R$ to $-\rho$ are equal to each other and to the integral we are after.
Now $f(z)$ has one pole inside the γ at $z = i$ and the value of the residue there is

$$\lim_{z\to i}(z-i)f(z) = \frac{e^{i\pi/3}}{2i}.$$

Thus by the residue Theorem

$$\begin{aligned}\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx &= \frac{1}{2}\,\mathrm{Re}\left(\int_\gamma f(z)\,\mathrm{d}z\right)\\ &= \frac{1}{2}\,\mathrm{Re}(2\pi i\frac{e^{\pi i/3}}{2i})\\ &= \frac{\sqrt{3}\pi}{4}.\end{aligned}$$

5. (15pts) Let E be a closed bounded subset of the complex plane. Prove that any complex function $f(z)$, holomorphic on the interior of E, achieves its maximum on the boundary of E.

Solution:
As E is closed and bounded it is compact. Thus f achieves its maximum on E. Hence it suffices to prove that if f achieves its maximum on U, the interior of E, then $f(z)$ is locally constant. In particular we may assume that U is connected, so that U is a region.
Suppose that $f(z)$ achieves its maximum at $a \in U$. Pick a small disc, centered at a, contained in U. Replacing U by this disc, we may assume that U is a disc. Let γ be the boundary of the disc. By Cauchy's Integral Formula

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} \, \mathrm{d}z.$$

Taking absolute values and parametrising γ in the usual way, we have

$$\begin{aligned} |f(a)| &= \frac{1}{2\pi} \left| \int_0^{2\pi} \frac{f(a + re^{i\theta})}{re^{i\theta}} ire^{i\theta} \, \mathrm{d}\theta \right| \\ &\leq \frac{1}{2\pi} M 2\pi = M, \end{aligned}$$

where M is the maximum value of $|f(z)|$ on γ. Since $M \leq |f(a)|$, we must have $M = |f(a)|$ and in fact $|f(z)|$ must be constant on γ. As γ is arbtirary, it follows easily that $|f(z)|$ is constant on U. But then $f(z)$ must also be constant.

6. (10pts) State and prove Schwartz's Lemma.

Solution:
Let $f(z)$ be a holomorphic function from the unit disc to the unit disc. Suppose that $f(0) = 0$ and $|f'(0)| \leq 1$. Then $|f(z)| \leq |z|$, for all $|z| < 1$ with equality if and only if $f(z) = \alpha z$, $|\alpha| = 1$.
Let

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0. \end{cases}$$

Then $g(z)$ is a holomorphic function. Note that $|g(z)| \leq 1/r$, for $|z| \geq r$. By question (5), $|g(z)| \leq 1/r$, and letting r tend to one, $|g(z)| \leq 1$. Thus $|f(z)| \leq |z|$. Suppose that we have equality. Then $|g(z)|$ achieves its maximum and, again by (5), $g(z) = \alpha$ is constant. But then $f(z) = \alpha z$.

Bonus Challenge Problems

7. (10pts) State and prove the Casorati-Weierstrass Theorem.

Solution:

Let $f(z)$ be a function, holomorphic on a region U, except at a, where $f(z)$ has an isolated essential singularity. Then for every $w \in \mathbb{C}$, and for every $\epsilon > 0$ and $\delta > 0$, there is a $|z-a| < \delta$ such that $|f(z)-w| < \epsilon$.

Suppose not. Consider $g(z) = \frac{f(z)-w}{z-a}$. As the numerator of $g(z)$ is bounded away from zero, in a neighbourhood of a, it follows that

$$\lim_{z \to a} g(z) = \infty.$$

But then $g(z)$ has a pole at a, so that we may write

$$g(z) = \frac{h(z)}{(z-a)^h},$$

where $h(z)$ is holomorphic in a neighbourhood of a. Solving for $f(z)$ we have

$$f(z) = \frac{g(z)}{(z-a)^{h-1}} + w.$$

But then $f(z)$ has a pole at a, a contradiction.

8. (10pts) State and prove Hurwitz's Theorem.

Solution:
Let $f_1, f_2, \ldots$ be a sequence of nowhere zero, holomorphic functions defined on a region U, which converges, uniformly on compact subsets, to $f(z)$.
If $f(z)$ is zero somewhere on U, then it is constant.
Suppose that $f(z)$ is not constant. Then the zeroes of $f(z)$ are isolated. Pick a point $a \in U$. Pick a small closed disc centred at a, inside which $f(z)$ has no zeroes, except possibly at a. Let γ be the circle bounding this disc. Then $1/f(z)$ is not zero on this circle, so that $1/f(z)$ is bounded away from zero. Then

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f_k'(z)}{f_k(z)} \, \mathrm{d}z = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, \mathrm{d}z.$$

But the LHS counts the number of zeroes of $f_k(z)$ in the disc, of which there are none, and the RHS counts the number of zeroes of $f(z)$. It follows that $f(z)$ has no zeroes in the disc. As a is arbitrary, we are done.

9. (10pts) Express $\sin \pi z$ as an infinite product.

Solution:
Consider the function

$$p(z) = z \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}.$$

Taking logs, as the series

$$\sum \frac{1}{n^2},$$

converges it follows from the Weierstrass Theorem, that the product is uniformly convergent on compact subsets, so that $p(z)$ is an entire holomorphic function. Moreover the ratio

$$\frac{\sin \pi z}{p(z)}$$

is nowhere zero, as the zeroes of the top and bottom cancel, so that we may write

$$\sin \pi z = e^{g(z)} p(z),$$

for some entire function $g(z)$. It remains to determine $g(z)$. Taking the logarithmic derivative, we have

$$\pi \cot \pi z = g'(z) + \frac{1}{z} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{n} + \frac{1}{z-n}\right).$$

But we have already seen that $g'(z) = 0$, so that $g(z)$ is constant. As

$$\lim_{z \to 0} \frac{\sin \pi z}{z} = \pi,$$

it follows that $e^{g(z)} = \pi$.
Thus

$$\sin \pi z = \pi z \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}.$$