PRACTICE PROBLEMS FOR THE FINAL

Here are a slew of practice problems for the final culled from old exams:

1. Let P_2 be the vector space of polynomials of degree at most 2. Let

$$\mathcal{B} = \{\, 1, (t-2)^2, t^2 + t \,\}.$$

(a) Show that $\mathcal{B}$ is a basis of P_2.
(b) Write $5t^2 + 5$ as a linear combination of the elements of $\mathcal{B}$.

2. Let A be the matrix:

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{pmatrix}$$

(a) Determine the eigenvalues of A.
(b) Find a basis for each eigenspace of A.
(c) Diagonalise A.

3. Let W be the subspace of $\mathbb{R}^4$ spanned by

$$\vec{x}_1 = (1, -3, 0, 1) \qquad \text{and} \qquad \vec{x}_2 = (5, -5, -1, 2).$$

(a) Use the Gram-Schmidt process to find an orthonormal basis for W.
(b) Find the projection of

$$\vec{y} = (1, 2, 1, 4)$$

onto W.
(c) Find the distance from $\vec{y}$ to W.
(d) Find a basis for $W^{\perp}$.

4. Fully justify each answer.
(a) Show that the set of eigenvalues of a square matrix A is the same as the eigenvalues of the matrix A^T.
(b) Suppose that A is a square matrix with distinct eigenvalues λ_1 and λ_2. Suppose that $\vec{x}_1$ and $\vec{x}_2$ are (non-zero) eigenvectors with eigenvalues λ_1 and λ_2. Show that $\vec{x}_1$ and $\vec{x}_2$ are linearly independent.

5. Orthogonally diagonalise the following symmetric matrix

$$\begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

6. Find a least squares solution to $A\vec{x} = \vec{b}$ where:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 2 \\ 5 \\ 6 \\ 6 \end{pmatrix}.$$

Is this solution unique?

7. Suppose that V is a vector space with subspaces U and W. Justify your answer to the following by providing a proof or a counterexample:

(a) Is

$$U \cap W = \{\, v \in V \mid v \in U \text{ and } v \in W \,\}$$

a subspace of V?

(b) Is

$$U \cup W = \{\, v \in V \mid v \in U \text{ or } v \in W \,\}$$

a subspace of V?

8. As always, justify your answer.

(a) Is it possible that all solutions to a homogeneous system of 10 equations with 12 unknowns are multiples of all single non-zero vector?

(b) Is it possible for a system of 6 equations with 5 unknowns to have a unique solution for a fixed right hand side of constants.

(c) Show that if A is diagonalisable and invertible, then so is A^{-1}.

9. (a) Let

$$\mathcal{D} = \{\, f(x) \in P_n \mid f'(0) = 0 \,\}$$

denote the subset of the polynomials of degree at most n whose derivative at zero is 0. Verify that $\mathcal{D}$ is a linear subspace of P_n.

(b) Suppose

$$A = \begin{pmatrix} 2 & -1 & 0 & 3 \\ -1 & 0 & 3 & 2 \end{pmatrix}.$$

Find a basis for $\text{Col}(A)$ and $\text{Nul}(A)$. What is $\text{rank}(A)$?

(c) Suppose that A is an $m \times n$ matrix with $m < n$. Suppose $\text{rank}(A) < n$. Is it possible that the columns of A span $\mathbb{R}^m$? Why or why not?

(d) Suppose that

$$H = \left\{ \begin{pmatrix} a + 2b + 3c \\ a + 2b + 3c \\ a + 2b + 3c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

Find a basis for H.

10. (a) Suppose

$$A = \begin{pmatrix} 3 & -3 & 5 \\ 0 & 4 & -3 \\ 0 & 2 & -1 \end{pmatrix}.$$

Find the eigenvalues of A with multiplicity.
(b) Diagonalise the matrix A from (a).
(c) Suppose B has eigenvalues 2 and 3 with corresponding eigenvectors $\vec{x}$ and $\vec{y}$ respectively. Suppose $\vec{z} = 10\vec{x} + 2\vec{y}$. Compute $B^{100}\vec{z}$. You may leave your answer in terms of $\vec{x}$ and $\vec{y}$.
11. (a) Use the Gram-Schmidt process to make

$$\mathcal{B} = \{\, (1, -1), (2, 3) \,\}$$

into an orthogonal basis of $\mathbb{R}^2$.
(b) Find the least squares solution to $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 7 \end{pmatrix}.$$

12. For each statement, mark it true or false. If it is false give a (counter)example. If it is true give a reason - if the reason is a theorem, state the theorem, otherwise give a brief proof. No credit for answers without a correct reason or example. Unless explicitly stated, no assumptions are made on the dimensions of the matrices.
(a) If A has n different eigenvectors, then A is diagonalisable.
(b) If $AP = PD$, where D is diagonal, then the columns of P are eigenvectors of A.
(c) If λ is an eigenvalue of A then λ^{100} is an eigenvalue of A^{100}.
(d) An orthogonal matrix has orthonormal rows.
(e) If AB is invertible and A and B are square then A is invertible.
(f) If $\vec{x}_0$ is the least squares solution to $A\vec{x} = \vec{b}$ then $\vec{b}_0 = A\vec{x}_0$ is the closest vector in $\mathrm{Col}(A)$ to $\vec{b}$.
13. (a) Suppose

$$A^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 2 & 3 & 3 \end{pmatrix}.$$

Solve $A\vec{x} = \vec{b}$.
(b) Define eigenvalue.
(c) Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the linear function that reflects across the x_1-axis and then in the line $x_1 = x_2$. Find the matrix associated to f.

14. Find the general solution to

$$\begin{aligned} x_1 \quad\quad\; + 5x_3 + \;\; 6x_4 &= 6 \\ x_1 + \;\; x_2 \quad\quad\quad + \;\; 2x_4 &= 1 \\ 3x_1 + 2x_2 + 6x_3 + 11x_4 &= 11. \end{aligned}$$

15. A linear function $f\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ is defined by

$$f(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + x_3, x_2 - x_3, x_1 + 3x_2, x_3 + x_4).$$

(a) Find the standard matrix for f.
(b) Find a basis for the nullspace of f.
16. For

$$A = \begin{pmatrix} 2 & 2 & 4 & 5 & 0 \\ 0 & 2 & 2 & 1 & 3 \\ 1 & 1 & 3 & 4 & 2 \\ 5 & 5 & 11 & 14 & 2 \end{pmatrix}$$

find the following.
(a) rank of A.
(b) a basis for the row space of A.
(c) a basis for all $\vec{b} \in \mathbb{R}^4$ for which $A\vec{x} = \vec{b}$ has a solution.
17. If the eigenvalues of A are 1, 2 and 3 what are the eigenvalues of A^{-1}? Give a brief reason for your answer.
18. For each statement, mark it True or False. If true, give a brief reason. If false, explain or give a counterexample. No credit if reason is wrong.
(a) If A and B are two 2×2 matrices, with A invertible and if $AB = 0$, then $B = 0$.
(b) If A is a 2×2 matrix which is diagonalisable, then A is symmetric.
(c) If the eigenvalues of a 3×3 matrix are 0, 1 and 2, then A is diagonalisable.
(d) Suppose that A and B are two square matrices, and B is obtained from A by row operations. Then every eigenvalue of A is an eigenvalue of B.
19. Let V be the plane in $\mathbb{R}^4$ spanned by the vectors $(2, 0, 1, 1)$ and $(1, 1, 0, 2)$. Find the vector in V closest to $\vec{y} = (3, 1, 5, 1)$.
20. Determine if the set of vectors in $\mathbb{R}^4$,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 0 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is linearly independent.

21. Suppose that you know the determinant of the matrix

$$A = \begin{pmatrix} 1 & a & 2 \\ 3 & b & 5 \\ -1 & c & -3 \end{pmatrix}$$

is 3 and $\vec{x} = (x_1, x_2, x_3)$ is a vector for which

$$A\vec{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Find x_2.

22. True or false? (+1 pt for correct answer, -1 pt for incorrect answer).

(a) If A is any matrix the system $A\vec{x} = \vec{0}$ must have at least one solution.

(b) If a square matrix is diagonalisable then its rows must be linearly independent.

(c) If V is a vector space and there is no set of n vectors which spans V then $\dim(V) > n$.

(d) If there is a linearly dependent set

$$\{\, \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \,\}$$

of vectors in V, then $\dim(V) < 4$.

(e) the function $f(x_1, x_2) = (x_1, x_2 + 1)$ is a linear function from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

(f) If A and B are two square matrices which are similar to each other, then they must have the same eigenvalues.

(g) If A and B are two square matrices which are similar to each other, then they must have the same eigenvectors.

(h) If A is a square matrix and 0 is an eigenvalue of $A - 2I$ then 2 is an eigenvalue of A.

(i) There is a linear function from $\mathbb{R}^3$ onto $\mathbb{R}^4$.

23. Give examples of 2×2 matrices A and B with the same characteristic polynomial but A is diagonalisable and B is not.

24. In this problem A is a square matrix. Give a very brief answer for each question.

(a) If A is not invertible, what number must be an eigenvalue of A?

(b) If $\dim \mathrm{Nul}(A) = 1$, what is the rank of A?

(c) If A is invertible what is the row reduced echelon form of A?

(d) If A is not invertible, find $\det A$.

(e) If $A^2 = 0$, show that A is not invertible.

25. If

$$\{\, \vec{v}_1, \vec{v}_2, \vec{v}_3 \,\}$$

is a linearly independent set of vectors in a vector space and $\vec{v}_4$ is a vector in V which is not in the span of

$$\{\,\vec{v}_1, \vec{v}_2, \vec{v}_3\,\},$$

show (carefully) that

$$\{\,\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\,\},$$

is a linearly independent set.

26. Find all eigenvalues and eigenvectors of

$$\begin{pmatrix} 9 & -2 \\ 2 & 5 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{pmatrix}.$$

27. The matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{pmatrix}$$

has one eigenvalue equal to -2. Diagonalise A.

28. Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Find a 3×3 diagonal matrix D and a 3×3 matrix orthogonal matrix U such that $A = UDU^{-1}$. Compute A^{10}.

29. Let $\vec{u} = (-1, 0, 1, 1)$ and $\vec{v} = (1, -1, -2, 0)$. Compute the length of $\vec{u}$ and $\vec{v}$.

30. Let $\vec{u}_1 = (-1, 3, 1, 1)$, $\vec{u}_2 = (6, -8, -2, -4)$ and $\vec{u}_3 = (6, 3, 6, -3)$. Let

$$W = \text{Span}\{\,\vec{u}_1, \vec{u}_2, \vec{u}_3\,\}$$

and let $\vec{y} = (1, 0, 0, 1)$.

(a) Find an orthogonal basis of W.

(b) Find the orthogonal projection $\mathbb{P}({}_W\vec{y})$

(c) Find the distance from $\vec{y}$ to W.

(d) Decompose the vector $\vec{y}$ as follows: $\vec{y} = \vec{y}_0 + \vec{y}_1$, where $\vec{y}_0 \in W$ and $\vec{y}_1$ is orthogonal to W.

31. Let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}.$$

(a) Find the orthogonal projection of $\vec{b}$ onto Col(A).

(b) Find the least squares solution $\vec{x}_0$ such that

$$\|\vec{b} - A\vec{x}_0\| \le \|\vec{b} - A\vec{x}\| \qquad \text{for all} \qquad \vec{x} \in \mathbb{R}^3.$$

(c) Find a basis for the orthogonal complement $\text{Col}(A)^{\perp}$.
32. Let

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\},$$

be three non-zero pairwise orthogonal vectors in $\mathbb{R}^4$. Prove that

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

is a linearly independent set.
33. Let W be the subspace spanned by $\vec{u}_1 = (1, -4, 0, 1)$ and $\vec{u}_2 = (7, -7, -4, 1)$. Find an orthogonal basis for W by performing the Gram-Schmidt process to these vectors. Find a basis for $W^{\perp}$.
34. True or false:
(a) If the matrices A and B are similar, that is, $A = PBP^{-1}$ for some invertible matrix P, then A and B have the same set of eigenvalues.
(b) Let $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ be three vectors in $\mathbb{R}^3$. Then they are linearly independent if and only if they are pairwise orthogonal.
(c) If $\det A = 0$ then 0 is an eigenvalue of A.
(d) The nullspace of an $m \times n$ matrix A consists of all vectors in $\mathbb{R}^n$ that are orthogonal to any vectors in the column space of A.
(e) Let B be a 6×8 matrix with $\dim \text{Nul}(B) = 3$. Then $\text{rank}(B) = 3$.
(f) A is diagonalisable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.
(g) Any invertible matrix is diagonalisable.
(h) Any upper triangular square matrix is diagonalisable.
(i) The inverse of a diagonalisable matrix is diagonalisable.
(j) An orthogonal matrix is orthogonally diagonalisable.
(k) Any three different eigenvectors of a matrix A corresponding to three different eigenvalues must be linearly independent.
(l) If λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2.
(m) If the columns of A are linearly independent, the equation $A\vec{x} = \vec{b}$ has exactly one least squares solution.
(n) If a square matrix has orthonormal columns then it has orthonormal rows.
(o) If a set

$$S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$$

has the property that $\vec{u}_i \cdot \vec{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.
(p) If A is a symmetric matrix and $A\vec{x} = 2\vec{x}$, $A\vec{y} = 3\vec{y}$ then $\vec{x} \cdot \vec{y} = 0$.
35. True or false:
(a) The row space of A is the column space of A^T.
(b) The inverse of an invertible $n \times n$ matrix A can be found by row reducing the augmented matrix $[A|I_n]$.

(c) If $\mathcal{B}$ and $\mathcal{C}$ are two bases of a vector space V then $\mathcal{B}$ and $\mathcal{C}$ have the same number of elements.
(d) Two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal if and only if their dot product is greater than or equal to $\vec{0}$.
(e) If $\mathcal{B}$ is a basis of a vector space V and $\mathcal{C}$ is a basis for a linear subspace then each vector in $\mathcal{C}$ can be written as a linear combination of the vectors in $\mathcal{B}$.
(f) The area of the parallelogram with vertices $(0, 0)$, $(1, 0)$, $(2, 3)$ and $(3, 3)$ is 9.
(g) If $m < n$ then the columns of an $m \times n$ matrix A could span $\mathbb{R}^n$.
(h) If A and B are row equivalent then $\text{Row}(A) = \text{Row}(B)$.
(i) If the second column of a matrix is a pivot column then x_2 is not a free variable.
(j) The determinant of a matrix is equal to the determinant of its transpose.
(k) A matrix with n distinct eigenvalues is diagonalisable.
(l) If A and B are row equivalent matrices then A and B have the same column space.
(m) A linearly independent set of vectors in $\mathbb{R}^n$ containing n vectors is a basis of $\mathbb{R}^n$.
(n) A set of more than n vectors in $\mathbb{R}^n$ that spans $\mathbb{R}^n$ is a basis of $\mathbb{R}^n$.
(o) A set of vectors in $\mathbb{R}^n$ that spans $\mathbb{R}^n$ contains a basis of $\mathbb{R}^n$.
(p) If V is a k-dimensional vector space and S is a set of $k+1$ vectors in V then S contains a basis of V.
(q) The Gram-Schmidt algorithm returns an orthogonal basis for a given subspace.
(r) If $\det(A) = d$ then $\det(kA) = k^n d$.
(s) If A is a 4×4 matrix and the null space of A is a plane in $\mathbb{R}^4$ then the column space of A has a basis with two elements.
(t) If $\vec{v}$ is a non-zero vector in V then

$$\frac{\vec{v}}{||\vec{v}||}$$

is a unit vector in the direction of $\vec{v}$.
36. Let

$$A = \begin{pmatrix} 5 & -1 & 1/2 \\ 4 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

(a) Calculate $\det(A)$.
(b) Find A^{-1}.
(c) Determine the span of the columns of A.
(d) Find the characteristic polynomial of A.

(e) Find the eigenvalue(s) of A.
(f) For each eigenvalue λ you found in part (e) find a basis for the associated eigenspace E_λ.
(g) If possible, find a matrix P and a diagonal matrix D such that $P^{-1}AP = D$. If not possible, explain why not.

37. Let

$$B = \begin{pmatrix} 1 & 2 & -3 & -4 & 5 & 6 \\ 0 & 3 & -4 & 1 & 9 & 10 \\ 0 & 2 & -1 & 0 & 8 & -2 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 4 & 0 & 0 & 8 & -3 \\ 0 & 3 & 0 & 0 & -4 & 2 \end{pmatrix}$$

(a) Calculate the determinant of B by choosing clever rows and columns along which to expand.
(b) What is the dimension of the row space of B? Justify your answer.
(c) What is the dimension of the column space of B? Justify your answer.

38. Please prove two of the following. Indicate clearly which two you would like graded.
(a) Assuming that A is an invertible $n \times n$ matrix, show that the homogeneous equation $A\vec{x} = \vec{0}$ has only the trivial solution without appealing to the Invertible Matrix Theorem. Then show that $A\vec{x} = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^n$.
(b) If V is a vector space and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are vectors in V prove that

$$\text{Span}\{\,\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\,\}$$

is a linear subspace of V.
(c) Let A be an $m \times n$ matrix. Prove that every vector in $\text{Nul}(A)$ is in the orthogonal complement of $\text{Row}(A)$.