Answers to Math 20F Final Practice Problems

December 12, 2014

Warning: this document contains only *answers* to the practice problems. It does not contain full solutions. Moreover, for problems where the answer is, effectively, the entire worked out solution (such as proofs, or justification for true/false questions), the answer has either been omitted or shortened. (This means, for example, that if you look at the justification for a true/false answer, you may see a very kurt reason, but on an exam, you would have to explain it in more detail.)

1. (a) Show $\mathscr{B}$ spans P_2 and is linearly independent.

   (b) $5t^2 + 5 = 1(1) + 1(t-2)^2 + 4(t^2+t)$.

2. (a) $\lambda = 2$ (multiplicity 1), $\lambda = 3$ (multiplicity 2).

   (b) $\lambda = 2$: $\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$. $\lambda = 3$: $\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

   (c) $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$; $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

3. (a) $\left\{\begin{bmatrix} 1/\sqrt{11} \\ -3/\sqrt{11} \\ 0 \\ 1/\sqrt{11} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ -1/\sqrt{11} \\ 0 \end{bmatrix}\right\}$.

   (b) $\begin{bmatrix} 1 \\ 7/11 \\ -4/11 \\ -1/11 \end{bmatrix}$.

(c) $\frac{15\sqrt{11}}{11}$.

(d) $\left\{\begin{bmatrix} 3/10 \\ 1/10 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/10 \\ 3/10 \\ 0 \\ 1 \end{bmatrix}\right\}$.

4. (a) $(A - \lambda I)^{\mathsf{T}} = A^{\mathsf{T}} - \lambda I$

   (b) Consider $A(c_1\vec{x}_1 + c_2\vec{x}_2)$.

5. $P = \begin{bmatrix} -2/3 & -1/\sqrt{5} & 4\sqrt{5}/15 \\ -1/3 & 2/\sqrt{5} & 2\sqrt{5}/15 \\ 2/3 & 0 & \sqrt{5}/3 \end{bmatrix}$; $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

6. $\vec{x}_0 = \begin{bmatrix} 1/3 \\ 14/3 \\ -5/3 \end{bmatrix}$; yes.

7. (a) Yes. Usual argument.

   (b) No. Take two distinct lines through the origin in $\mathbb{R}^2$. (On an exam you need to write more than this. Actually come up with two different lines, and show which subspace property their union violates.)

8. (a) No. Rank-nullity $\implies$ smallest dim for nullspace is 2.

   (b) Yes. 5×5 system with unique solution, then add equation $0 = 0$.

   (c) Use determinants or invertible matrix theorem to use that D^{-1} exists.

9. (a) Usual argument.

   (b) Basis for Col(A) : $\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right\}$. Basis for Nul($A$) : $\left\{\begin{bmatrix} 3 \\ 6 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 0 \\ 1 \end{bmatrix}\right\}$.

   A has rank 2.

   (c) Yes. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(d) $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

10. (a) $\lambda = 1$ (multiplicity 1), $\lambda = 2$ (multiplicity 1), $\lambda = 3$ (multiplicity 1).

    (b) $P = \begin{bmatrix}-1 & -1 & 1\\ 1 & 3 & 0\\ 1 & 2 & 0\end{bmatrix}$; $D = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$.

    (c) $10 \cdot 2^{100}\vec{x} + 2 \cdot 3^{100}\vec{y}$.

11. (a) $\left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}5/2\\5/2\end{bmatrix}\right\}$.

    (b) $\begin{bmatrix}8\\5/2\\21/4\end{bmatrix}$

12. (a) False. (Take A to be $(n+1) \times (n+1)$.)

    (b) False. (Take $P = 0$.)

    (c) True.

    (d) True. $A^{\mathsf{T}} = A^{-1}$ is also orthogonal.*

    (e) True. $\det(AB) = \det(A)\det(B)$.

    (f) True.

13. (a) $\vec{x} = A^{-1}\vec{b}$.†

    (b) Usual definition.

    (c) $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$.

14. $\begin{bmatrix}-9\\10\\3\\0\end{bmatrix} - x_4\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$.

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*This is true for *our* class, since we require an orthogonal matrix to be square. However, for the class in which this problem was posed, the answer was probably false.

†On the exam which had this question, likely $\vec{b}$ was given explicitly, so one would have to actually do a bit of computation.

15. (a) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

(b) $\left\{ \begin{bmatrix} 3 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

16. (a) 3

(b) $\left\{ \begin{bmatrix} 1 & 0 & 0 & 1/2 & -7/2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & -1 & -1/2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3/2 & 2 \end{bmatrix} \right\}$.[‡]

(c) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \\ 11 \end{bmatrix} \right\}$.

17. 1, 1/2, and 1/3; multiply both sides of $A\vec{x} = \lambda\vec{x}$ by A^{-1}.

18. (a) True. $\boldsymbol{B} = A^{-1}0$.

(b) False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

(c) True.

(d) False. $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, $\boldsymbol{B} = \boldsymbol{I}_{2\times 2}$.

19. $\begin{bmatrix} 21/5 \\ -3/5 \\ 12/5 \\ 6/5 \end{bmatrix}$.

20. The vectors are linearly *dependent*.

21. $x_2 = 10/3$.

22. (a) True.

(b) False.

[‡]There are multiple correct answers; this basis is obtained from the *reduced* row echelon form.

(c) True.

(d) False.

(e) False.

(f) True.

(g) False.

(h) True.

(i) False.

23. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

24. (a) 0.

    (b) $n - 1$.

    (c) $I_{n \times n}$.

    (d) 0.

    (e) Consider $\det(A)$.

25. Start with $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$. Show $c_4 = 0$, then conclude that $c_1 = c_2 = c_3 = 0$. (On an exam you would *need* to fill in more details here.)

26. $\begin{bmatrix} 9 & -2 \\ 2 & 5 \end{bmatrix}$: eigenvalue 7, eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\begin{bmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{bmatrix}$: eigenvalues 3, 5, 10; eigenvectors $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} -7 \\ 14 \\ 11 \end{bmatrix}$.

27. $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

28. $U = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$; $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. $A^{10} = \begin{bmatrix} \frac{3^{10}+1}{2} & \frac{1-3^{10}}{2} & 0 \\ \frac{1-3^{10}}{2} & \frac{3^{10}+1}{2} & 0 \\ 0 & 0 & 2^{10} \end{bmatrix}$.

29. $\|\vec{u}\| = \sqrt{3}$; $\|\vec{v}\| = \sqrt{6}$.

30. (a) $\left\{\begin{bmatrix}-1\\3\\1\\1\end{bmatrix}, \begin{bmatrix}3\\1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\\3\\-1\end{bmatrix}\right\}$.

(b) $\begin{bmatrix}2/3\\1/3\\-1/3\\0\end{bmatrix}$.

(c) $\frac{2\sqrt{3}}{3}$.

(d) $\vec{y}_0 = \begin{bmatrix}2/3\\1/3\\-1/3\\0\end{bmatrix}$, $\vec{y}_1 = \begin{bmatrix}1/3\\-1/3\\1/3\\1\end{bmatrix}$.

31. (a) $\begin{bmatrix}3\\0\\-3\end{bmatrix}$.

(b) $\vec{x}_0 = \begin{bmatrix}2\\-1\end{bmatrix}$.

(c) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$.

32. Start with $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Take dot product and conclude that $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. (As before, you *need* to fill in more details here.)

33. Basis for W: $\left\{\begin{bmatrix}1\\-4\\0\\1\end{bmatrix}, \begin{bmatrix}5\\1\\-4\\-1\end{bmatrix}\right\}$. Basis for $W^\perp$: $\left\{\begin{bmatrix}16/21\\4/21\\1\\0\end{bmatrix}, \begin{bmatrix}1/7\\2/7\\0\\1\end{bmatrix}\right\}$.

34. (a) True.

(b) False.

(c) True.

(d) False.

(e) False.

(f) False.[§]

(g) False.

(h) False.

(i) True.[¶]

(j) False.

(k) True.

(l) True.

(m) True.

(n) True.

(o) False.

(p) True.

35. (a) True.

(b) True.

(c) True.

(d) False.

(e) True.

(f) False.

(g) False.

(h) True.

(i) True.

(j) True.

(k) False.[‖]

(l) False.

(m) True.

(n) False.

[§] The question as stated does not say that $\boldsymbol{D}$ is a *diagonal* matrix.

[¶] This assumes that the inverse exists.

[‖] Again, they did not give the dimensions of the matrix.

(o) True.

(p) False.

(q) True.

(r) False.[**]

(s) True.

(t) True.

36. (a) 27

(b) $\begin{bmatrix} 1/9 & 1/9 & 1/18 \\ -4/9 & 5/9 & -1/9 \\ 0 & 0 & 1/3 \end{bmatrix}$.

(c) $\mathbb{R}^3$.

(d) $\lambda^3 - 9\lambda^2 + 27\lambda - 27$.

(e) $\lambda = 3$.

(f) E_3: $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/4 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(g) Not diagonalizable.

37. (a) 17.

(b) 6. B is invertible, so pivot in every row.

(c) 6. B is invertible, so pivot in every column.

38. Make sure you fill in more details:

(a) Multiply both sides of $A\vec{x} = \vec{0}$ by A^{-1}.

(b) Usual argument.

(c) Use matrix form of dot product, i.e., $\vec{u} \cdot \vec{v} = \vec{u}^{\mathsf{T}}\vec{v}$, with columns of A.

[**] Once again, we don't know the dimensions of A.