FIRST MIDTERM
MATH 20F, UCSD, AUTUMN 14

You have 50 minutes. This test is closed book, closed notes, no calculators.

There are 7 problems, and the total number of points is 100. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Dissertation instructor:____________________

Dissertation Number+Time:__________________

Problem	Points	Score
1	20	
2	20	
3	10	
4	10	
5	15	
6	15	
7	10	
Total	100	

1. (20pts) (i) Show that the matrix

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 1 \\ -1 & 1 & -5 & 12 \end{pmatrix} \qquad \text{has echelon form} \qquad \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To get full credit for this problem, you **must** show your steps and explain what row operations you are using at each stage.

Solution:
We apply Gaussian elimination. We multiply the first row by -2 and 1 and add it to the second and third rows:

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 1 \\ -1 & 1 & -5 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & -3 & 15 \end{pmatrix}$$

Now multiply the second row by 3 and add it to the third row:

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & -3 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(ii) Identify the basic variables and the free variables of the linear system of question.

Solution:
The pivots are in the first and third columns. So x and z are the basic variables and the remaining variables y and w are the free variables.

2. (20pts)
(i) Find the general solution to the linear equations in parametric form:

$$\begin{aligned} x - y + 2z + 3w &= 0 \\ 2x - 2y + 5z + w &= 0 \\ -x + y - 5z + 12w &= 0 \end{aligned}$$

Solution: The coefficient matrix is the matrix A in question (1). We solve the equations from the echelon form using back substitution. By (1) (ii) y and w are the free variables.

$$z - 5w = 0 \qquad \text{so that} \qquad z = 5w.$$

Therefore

$$x - y + 2(5w) + 3w = 0 \qquad \text{so that} \qquad x = y - 13w.$$

The general form is

$$(x, y, z, w) = (y - 13w, y, 5w, w) = y(1, 1, 0, 0) + w(-13, 0, 5, 1).$$

(ii) Check that $(x, y, z, w) = (1, 1, -1, 1)$ is a solution to

$$\begin{aligned} x - y + 2z + 3w &= 1 \\ 2x - 2y + 5z + w &= -4 \\ -x + y - 5z + 12w &= 17. \end{aligned}$$

Solution:

$$\begin{aligned} 1 - 1 - 2 + 3 &= 1 \\ 2 - 2 - 5 + 1 &= -4 \\ -1 + 1 + 5 + 12 &= 17. \end{aligned}$$

(iii) Find the general solution to the linear equations, given in part (ii), in parametric form.

Solution:
The general solution is a sum of the particular solution $(1, 1, -1, 1)$ and the general solution to the homogeneous:

$$(x, y, z, w) = (1, 1, -1, 1) + y(1, 1, 0, 0) + w(-13, 0, 5, 1).$$

3. (10pts) (i) Show that the vectors $(1,0,0)$, $(10,2,0)$ and $(-15,3,1)$ span $\mathbb{R}^3$.

Solution:
Let A be the matrix whose columns are the vectors $(1,0,0)$, $(10,2,0)$ and $(-15,3,1)$. We want to show that we can always solve the equation $A\vec{x} = \vec{b}$. We apply elimination to

$$A = \begin{pmatrix} 1 & 10 & -15 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 10 & -15 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Every row contains a pivot. It follows that the augmented matrix contains no pivots in the last column and the equation $A\vec{x} = \vec{b}$ is consistent. Therefore the vectors $(1,0,0)$, $(10,2,0)$ and $(-15,3,1)$ span $\mathbb{R}^3$.

4. (10pts) Find the values of h for which the vectors $(-1,3,2)$, $(2,-6,-5)$ and $(1,h,1)$ in $\mathbb{R}^3$ are linearly dependent.

Solution: Let A be the matrix whose columns are the vectors $(-1,3,2)$, $(2,-6,-5)$ and $(1,h,1)$. We want to find those values of h such that $A\vec{x} = \vec{0}$ has a non-trivial solution. We apply elimination to

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -6 & h \\ 2 & -5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 \\ 3 & -6 & h \\ 2 & -5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 \\ 0 & 0 & h+3 \\ 0 & -1 & -1 \end{pmatrix}$$

so that

$$\to \begin{pmatrix} 1 & -2 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & h+3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & h+3 \end{pmatrix}.$$

There is a non-trivial solution if and only if there is a free variable. The only possibility is that the last variable is a free variable, that is, $h + 3 = 0$. So the vectors $(-1,3,2)$, $(2,-6,-5)$ and $(1,h,1)$ in $\mathbb{R}^3$ are linearly dependent if and only if $h = -3$.

5. (15pts) Find the inverse of

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}.$$

Solution:
We apply Gauss-Jordan elimination. We first form the super augmented matrix

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & -2 & 1 \end{array}\right)$$

Multiplying the last row by -1 completes the Gaussian elimination:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

The inverse is

$$A^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}.$$

6. (15pts) (i) Let f be the linear function

$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad \text{given by} \qquad (x, y, z) \longrightarrow (2x - y + 3z, x + 2y - z).$

Find a matrix A such that $f(\vec{x}) = A\vec{x}$.

Solution: $f(1,0,0)$=$(2,1)$, $f(0,1,0) = (-1,2)$ and $f(0,0,1) = (3,-1)$ and so

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$

(ii) Let g be the linear function $g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which represents rotation through $\pi/2$ about the origin. Find a matrix B such that $g(\vec{x}) = B\vec{x}$.

Solution: $g(1,0) = (0,1)$ and $g(0,1) = (-1,0)$ and so

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(iii) Find a matrix C such that $(g \circ f)(\vec{x}) = C\vec{x}$.

Solution: Composition of functions corresponds to matrix multiplication:

$$C = BA = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 & 1 \\ 2 & -1 & 3 \end{pmatrix}.$$

7. (10pts) Let h be the linear function $h\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $h(1,1) = (5,4)$ and $h(1,2) = (-1,0)$. Find a matrix D such that $h(\vec{x}) = D\vec{x}$.

Solution: $(0,1) = (1,2) - (1,1)$. By linearity

$$h(0,1) = h(1,2) - h(1,1) = (-1,0) - (5,4) = (-6,-4).$$

$(1,0) = (1,1) - (0,1)$. By linearity

$$h(1,0) = h(1,1) - h(0,1) = (5,4) - (-6,-4) = (11,8).$$

Therefore

$$D = \begin{pmatrix} 11 & -6 \\ 8 & -4 \end{pmatrix}.$$

We check:

$$\begin{pmatrix} 11 & -6 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 11 & -6 \\ 8 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$