8. The Inverse Matrix

Consider a very easy linear equation, something of the form

$$ax = b.$$

For example

$$2x = 3.$$

We solve this by multiplying both sides by the inverse of 2,

$$(1/2)2x = (1/2)3 \qquad \text{so that} \qquad x = 3/2.$$

In general, if $a \neq 0$ then we can play the same trick. Multiply both sides by the inverse and simplify

$$(1/a)ax = b/a \qquad \text{so that} \qquad x = a^{-1}b = b/a.$$

Now consider the matrix equation

$$A\vec{x} = \vec{b}$$

Wouldn't it be nice to play the same trick?

Definition 8.1. *Let A be a $n \times n$ square matrix. We say that A is invertible, and call $C = A^{-1}$ the inverse of A, if*

$$AC = CA = I_n.$$

Note that C is an $n \times n$ matrix. Let's suppose that C is the inverse of A. Multiply both sides of the equation above by C:

$$C(A\vec{x}) = C\vec{b}.$$

But

$$C(A\vec{x}) = (CA)\vec{x} = I_n\vec{x} = \vec{x}.$$

So

$$\vec{x} = C\vec{b} = A^{-1}\vec{b}.$$

Theorem 8.2. *Let A be an invertible matrix.*

Then the equation

$$A\vec{x} = \vec{b}$$

has the unique solution

$$\vec{x} = A^{-1}\vec{b}.$$

Notice for this to work the solution always exists and it is unique, so that A has to be a square matrix. If A has more rows than columns then sometimes we could find an inconsistent equation and if A has more columns than rows then sometimes we would have more than one solution.

In general it is computationally quite expensive to find the inverse of a matrix. However there are a couple of simple cases. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is 2×2 matrix then the inverse of A is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For this to make sense the number $ad - bc \neq 0$. We call $ad - bc$ the determinant, since it determines whether or not A has an inverse.

What is the inverse of

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}?$$

We can either use the formula or realise that this is the identity, so that it is its own inverse.

How about

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}?$$

This represents rotation through π, which is its own inverse. How about

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}?$$

This represents rotation through $\pi/2$ anticlockwise. The inverse is rotation through $\pi/2$ clockwise,

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proposition 8.3.

(1) If A is invertible then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(2) If A and B are invertible then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(3) If A is invertible then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Proof. We only prove (2). There are two ways to see this. We could just compute:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n.$$

Similarly the other way around. Then AB is invertible and $B^{-1}A^{-1}$ is the inverse.

Or we could think in terms of functions. AB represents the composition $f \circ g$, first do g then do f. To undo this, first undo f then undo g, which is represented by $B^{-1}A^{-1}$. □

We now introduce a general method to find the inverse of a matrix. There are two ways to see why this method works. First in term of elimination and solving linear equations.

Consider the matrix equation

$$AC = I_n.$$

Let's break this down into pieces, column by column of C. Let $\vec{c}_i$ be the ith column of C. When we multiply by A we should get the ith column of I_n. The ith column of I_n is the vector $\vec{b} = \vec{e}_i$. We want to solve the equation

$$A\vec{x} = \vec{b}.$$

Here $\vec{b} = \vec{e}_i$ is the ith column of I_n and the solution $\vec{x}$ is the ith column $\vec{c}_i$ of C. To solve this equation apply Gaussian elimination. Form the augmented matrix

$$B_i = \left(A \mid \vec{e}_i\right)$$

Now apply Gaussian elimination with a twist. Instead of stopping at echelon form only stop at reduced echelon form; this is called Gauss-Jordan elimination:

$$\left(I_n \mid \vec{c}_i\right).$$

The elimination is complete. If we solve these equations by back substitution we will see that $\vec{x} = \vec{c}_i$.

Here comes the clever bit. Note that the steps of the elimination are always the same independently of the last column. The steps only depend on the coefficient matrix A. So let's form a super-augmented matrix. Put all of the vectors $\vec{e}_i$ on the RHS in the obvious order. The RHS is then the identity matrix:

$$B = \left(A \mid I_n\right)$$

Now apply Gaussian-Jordan elimination. At the end the ith column on the left is the ith column of C. So what appears on the RHS at the end of the elimination is C:

$$\left(I_n \mid C\right).$$

Example 8.4. *Find the inverse of*

$$A = \begin{pmatrix} 1 & 3 & 4 \\ -1 & -4 & -2 \\ 2 & 3 & 15 \end{pmatrix}.$$

We apply Gaussian-Jordan elimination to the super-augmented matrix:

$$B = \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ -1 & -4 & -2 & 0 & 1 & 0 \\ 2 & 3 & 15 & 0 & 0 & 1 \end{array}\right).$$

We first eliminate the two entries in the first column. We multiply the first row by 1 and -2 and add it to the second and third rows:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ -1 & -4 & -2 & 0 & 1 & 0 \\ 2 & 3 & 15 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & -3 & 7 & -2 & 0 & 1 \end{array}\right).$$

Now multiply the second row by -1:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & -3 & 7 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & -3 & 7 & -2 & 0 & 1 \end{array}\right).$$

We now eliminate the last entry in the second column. We multiply the second row by 3 and add it to the third row:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & -3 & 7 & -2 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right).$$

If we were simply applying Gaussian elimination, we would stop here. For Gauss-Jordan we have to create three more zeroes. We eliminate the -2 and 4 in the third column, second and first row. We multiply the third row by 2 and -4 and add it to the second and first row:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 21 & 12 & -4 \\ 0 & 1 & 0 & -11 & -7 & 2 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right).$$

Finally we create a zero in the first row second column by multiplying the second row by -3 and adding it to the first row:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 21 & 12 & -4 \\ 0 & 1 & 0 & -11 & -7 & 2 \\ 0 & 0 & 1 & -5 & -3 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 54 & 33 & -10 \\ 0 & 1 & 0 & -11 & 5 & 2 \\ 0 & 0 & 1 & -5 & -3 & 1 \\ . \end{array}\right)$$

Let's check that

$$C = \begin{pmatrix} 54 & 33 & -10 \\ -11 & 5 & 2 \\ -5 & -3 & 1 \\ . \end{pmatrix}$$

is the inverse of A. For example consider the product CA. If we take the third row of C and multiply by the second column of A are supposed to get zero:

$$-5 \cdot 3 + (-3 \cdot -4) + (1 \cdot 3) = -15 + 12 + 3 = 0.$$

Now consider the matrix product AC. If we take the last row of A and the last column of C we are supposed to get 1:

$$2 \cdot -10 + 3 \cdot 2 + 15 \cdot 1 = -20 + 6 + 15 = 1.$$