7. Matrix multiplication

Let A be an $m \times n$ matrix, so that A has m rows and n columns. It is convenient to be able to refer to the entries of A. The notation

$$A = (a_{ij}),$$

means that the (i, j)-entry, that is, the entry in the ith row and jth column, is a_{ij}.

If A and B are both $m \times n$ matrices, so that they have the same shape, we can add them. If $A = (a_{ij})$ and $B = (b_{ij})$ and

$$C = (c_{ij}) = A + B$$

is the sum then C is an $m \times n$ matrix then $c_{ij} = a_{ij} + b_{ij}$. A simple example will hopefully make this clear:

Example 7.1. *Let*

$$A = \begin{pmatrix} -1 & 2 & -3 \\ -3 & 0 & 5 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 1 & 3 \\ -3 & 7 & 4 \end{pmatrix}$$

Then

$$C = A + B = \begin{pmatrix} -1+2 & 2+1 & -3+3 \\ -3+-3 & 0+7 & 5+4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ -6 & 7 & 9 \end{pmatrix}.$$

Note that

$$A + B = B + A,$$

since

$$a_{ij} + b_{ij} = b_{ij} + a_{ij}.$$

Note that the zero matrix O of shape $m \times n$ acts as the additive zero,

$$A + O = O + A = A.$$

We can also multiply a matrix A by a scalar λ. If $C = \lambda A$ then C has the same shape as A and $c_{ij} = \lambda a_{ij}$.

Example 7.2. *Let*

$$A = \begin{pmatrix} -1 & 2 & -3 \\ -3 & 0 & 5 \end{pmatrix} \qquad \text{and} \qquad \lambda = 3.$$

$$C = 3A = \begin{pmatrix} 3\cdot -1 & 3\cdot 2 & 3\cdot -3 \\ 3\cdot -3 & 3\cdot 0 & 3\cdot 5 \end{pmatrix} = \begin{pmatrix} -3 & 6 & -9 \\ -9 & 0 & 15 \end{pmatrix}.$$

Note that scalar multiplication distributes over matrix addition:

$$\lambda(A + B) = \lambda A + \lambda B.$$

How should we multiply matrices? Well matrices correspond to functions. Now we can multiply functions together but if you multiply to linear functions together, you almost never get a linear function. Take the identity function

$$g\colon \mathbb{R} \longrightarrow \mathbb{R} \qquad \text{given by} \qquad x \longrightarrow x$$

If you naively multiply g with itself you get

$$\mathbb{R} \longrightarrow \mathbb{R} \qquad \text{given by} \qquad x \longrightarrow x^2$$

which is not linear.

One can also compose functions. Consider the linear function rotation through $\pi/2$. If you compose this with itself you get rotation through π, another linear function. Or if you compose with the linear function reflection in the x-axis you get the linear function reflection in the line $y = -x$.

In general if we are given

$$g\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad \text{and} \qquad f\colon \mathbb{R}^m \longrightarrow \mathbb{R}^p$$

then you can compose g with f to get

$$f \circ g\colon \mathbb{R}^n \longrightarrow \mathbb{R}^p \qquad \text{given by} \qquad \vec{x} \longrightarrow f(g(\vec{x})).$$

This suggests that given an $m \times n$ matrix B, corresponding to g and an $p \times m$ matrix A corresponding to f the matrix product $C = AB$ should be a $p \times n$ matrix.

We define the matrix product the same way we define multiplying a vector by a matrix. We pair rows of A with columns of B. The (i, j) entry c_{ij} of the product $C = AB$ is the sum of the products of the ith row of A with the jth column of B,

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}nj = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Let's do some examples. Suppose that

$$A = \begin{pmatrix} -1 & 2 & -3 \\ -3 & 0 & 5 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & -3 \\ 1 & 7 \\ 3 & 4 \end{pmatrix}$$

A is a 2×3 matrix, B is a 3×2 matrix and the product is a 2×2 matrix

$$\begin{pmatrix} -1 & 2 & -3 \\ -3 & 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 7 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -1 \cdot 2 + 2 \cdot 1 + (-3) \cdot 3 & -1 \cdot -3 + 2 \cdot 7 + (-3) \cdot 4 \\ -3 \cdot 2 + 0 \cdot 1 + 5 \cdot 3 & -3 \cdot -3 + 0 \cdot 7 + 5 \cdot 4 \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 5 \\ 9 & 29 \end{pmatrix}.$$

Recall that

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

represents rotation through $\pi/2$. Let's compute A^2,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

This represents rotation through π, as expected. Recall that

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

represents reflection in the x-axis. Let's compute BA,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

What function does this represent?

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}.$$

This sends $(1,0)$ to $(0,-1)$ and $(0,1)$ to $(-1,0)$, reflection in the line $y = -x$.

Is matrix multiplication commutative, that is, given two matrices A and B does the order of multiplication matter, does

$$AB = BA?$$

Suppose that

$$A = \begin{pmatrix} -1 & 2 \\ -3 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & 1 & 3 \\ -3 & 7 & 4 \end{pmatrix}$$

Then A is a 2×2 matrix and B is a 2×3. Then we can multiply A by B to get a 2×3 matrix AB:

$$\begin{pmatrix} -1 & 2 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ -3 & 7 & 4 \end{pmatrix} = \begin{pmatrix} -8 & 13 & 5 \\ -6 & -3 & -9 \end{pmatrix}$$

However the product BA does not even make sense. B is a 2×3 and A is 2×2 matrix. It is even clearer if you think in terms of functions. A corresponds to a linear function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and B corresponds to a linear function $g\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$. It makes sense to compose g with f, $f \circ g\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$. It does not make sense to compose f with g.

Now consider the matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

A is a 1×3 matrix and B is a 3×1 matrix. The product is a 1×1 matrix:

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \end{pmatrix}$$

The product BA in the other order makes sense but it is a 3×3 matrix:

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 3 & 6 & 9 \end{pmatrix}.$$

So $AB \neq BA$, even though both sides make sense. In terms of functions, the composition of $g\colon \mathbb{R} \longrightarrow \mathbb{R}^3$ and $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a function $f \circ g\colon \mathbb{R} \longrightarrow \mathbb{R}$ and the composition the other way is a function $g \circ f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

Finally suppose A and B are both square matrices, let's say 2×2:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then both AB and BA are 2×2 matrices. We have

$$AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

So $AB \neq BA$ even though both sides have the same shape. In terms of functions the product AB represents the composition of reflection in the x-axis and then rotation through $\pi/2$ and the second represents rotation through $\pi/2$ and then reflection in the x-axis. The first is reflection in the line $y = x$ and the second is reflection in the line $y = -x$. So matrix multiplication is not commutative.

Given a matrix A the transpose of A, denoted A^t, is obtained from A by switching the rows and columns. If $A = (a_{ij})$ has shape $m \times n$ then $A^t = B = (b_{ij})$ has shape $n \times m$. We have $b_{ij} = a_{ji}$.

If

$$A = \begin{pmatrix} -1 & 2 & -3 \\ -3 & 0 & 5 \end{pmatrix}$$

then A is a 2×3 matrix then the transpose

$$B = \begin{pmatrix} -1 & -3 \\ 2 & 0 \\ -3 & 5 \end{pmatrix}$$

is a 3×2 matrix.