30. Final review II

Example 30.1. *Suppose that b_1, b_2, b_3 and b_4 are real numbers.*

Show that there is exactly one polynomial $p(t)$ in the vector space P_3 of polynomials of degree at most 3 such that:

$$p(1) = b_1, \qquad p'(0) = b_2, \qquad \int_{-1}^{1} p(t)\,\mathrm{d}t = b_3, \qquad \textit{and} \qquad p(-1) = b_4.$$

A general polynomial of degree at most 3 looks like

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

We just write interpret the given conditions in terms of the coefficients a_0, a_1, a_2 and a_3.

$$p(1) = a_0 + a_1 + a_2 + a_3 \qquad \text{and so} \qquad a_0 + a_1 + a_2 + a_3 = b_1.$$

$$p'(t) = a_1 + 2a_2 t + 3a_3 t^2.$$

Therefore

$$p'(0) = a_1 + 2a_2 + 3a_3 \qquad \text{and so} \qquad a_1 + 2a_2 + 3a_3 = b_2.$$

$$\begin{aligned}\int_{-1}^{1} p(t)\,\mathrm{d}t &= \int_{-1}^{1} a_0 + a_1 t + a_2 t^2 + a_3 t^3\,\mathrm{d}t \\ &= \left[a_0 t + a_1 t^2/2 + a_2 t^3/3 + a_3 t^4/4\right]_{-1}^{1} \\ &= 2a_0 + 2a_3/3.\end{aligned}$$

Therefore

$$2a_1 + 2a_3/3 = b_3.$$

Finally

$$p(-1) = a_0 - a_1 + a_2 - a_3 \qquad \text{and so} \qquad a_0 - a_1 + a_2 - a_3 = b_3.$$

Thus we get the system of linear equations

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= b_1 \\ a_1 + 2a_2 + 3a_3 &= b_2 \\ 2a_1 + 2a_3/3 &= b_3 \\ a_0 - a_1 + a_2 - a_3 &= b_4. \end{aligned}$$

To make things more interesting we modify these equations to:

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= b_1 \\ a_0 + a_1 + 2a_2 + 3a_3 &= b_2 \\ a_1 + a_3/2 &= b_3 \\ a_0 - a_1 + a_2 - a_3 &= b_4. \end{aligned}$$

As usual we can rewrite this as $A\vec{a} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1/2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

We want to know that the equation $A\vec{a} = \vec{b}$ always has a unique solution. This is equivalent to saying that A is invertible.

There are two ways to check this. One is to apply Gaussian elimination and check we get four pivots:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1/2 \\ 1 & -1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1/2 \\ 1 & 1 & 2 & 3 \\ 1 & -1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 2 \\ 0 & -2 & 0 & -2 \end{pmatrix}$$

and so

$$\to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

There are four pivots and so A is invertible. In fact one can see directly that the equation $A\vec{x} = \vec{b}$ always has exactly one solution.

The other way to proceed is to compute the determinant. A is invertible if and only if $\det A \neq 0$:

$$\begin{aligned}
\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1/2 \\ 1 & -1 & 1 & -1 \end{vmatrix} &= -\begin{vmatrix} 0 & 1 & 0 & 1/2 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} + 1/2 \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{vmatrix} + 1/2 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{vmatrix} \\
&= -\begin{vmatrix} 0 & 0 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} \\
&= -2\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\
&= 2 - 1 = 1.
\end{aligned}$$

Example 30.2. *Let*

$$W = \{\, (a, b, c, d) \mid a + b = c + d, 2a + 2b = 3d \,\}.$$

Find a basis for $W^{\perp}$.

Implicit in this question is that W is a linear subspace. W is the set of solutions of the homogeneous equations

$$\begin{aligned} a + b - c - d &= 0 \\ 2a + 2b - 3d &= 0. \end{aligned}$$

Therefore if we put

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 2 & 0 & -3 \end{pmatrix}$$

then W is the null space of A. In particular W is a linear space. To find a basis of W, apply Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 2 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$$

The elimination is complete. a and c are basic variables, b and d are free variables.

$$c - d/2 = 0 \qquad \text{so that} \qquad c = d/2.$$

Therefore

$$a + b - d/2 - d = 0 \qquad \text{so that} \qquad a = -b - 3d/2.$$

The general solution is

$$(a, b, c, d) = (-b - 3d/2, b, d/2, d) = b(-1, 1, 0, 0) + d(-3/2, 0, 1/2, 1).$$

W is the span of $\vec{w}_1 = (1, -1, 0, 0)$ and $\vec{w}_2 = (-3, 0, 1, 2)$.

Let $\vec{v} = (p, q, r, s) \in W^\perp$ be a general vector. By definition $\vec{v}$ is orthogonal to every vector in W. As $(1, -1, 0, 0)$ and $(-3, 0, 1, 2)$ are a basis of W this is equivalent to requiring that $\vec{v}$ is orthogonal to both of these vectors:

$$\vec{v} \cdot \vec{w}_1 = 0 \qquad \text{and} \qquad \vec{v} \cdot \vec{w}_2 = 0.$$

We get a pair of homogeneous linear equations:

$$\begin{aligned} p - q &= 0 \\ -3p + r + 2s &= 0. \end{aligned}$$

We can make this into a matrix:

$$B = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -3 & 0 & 1 & 2 \end{pmatrix}.$$

$W^\perp$ is the null space of B. We can find a basis by Gaussian elimination, the usual way. But there is a much easier way.

W is nullspace of A. The nullspace of A is orthogonal to the row space of A.

$$W^\perp = \mathrm{Nul}(A)^\perp = \mathrm{Row}(A).$$

A basis for the row space of A is $(1, 1, -1, -1)$ and $(2, 2, 0, -3)$.

Example 30.3. *What are the eigenvalues of*

$$A = \begin{pmatrix} -2 & 3 & 1 & 1 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix}?$$

Since A is upper triangular, the eigenvalues are the entries on the main diagonal, -2, 1, 4 and 7.