28. Symmetric Matrices

Definition 28.1. *A matrix A is symmetric if $A^T = A$.*

Note that symmetric matrices are necessarily square.

Example 28.2.

$$\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

is symmetric but

$$\begin{pmatrix} 1 & -2 \\ 5 & 3 \end{pmatrix}$$

is not. The transpose is

$$\begin{pmatrix} 1 & 5 \\ -2 & 3 \end{pmatrix}.$$

Let's try to diagonalise a symmetric matrix:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

We look for the eigenvalues. If

$$A\vec{v} = \lambda\vec{v}$$

then $\vec{v}$ is an eigenvector and λ is an eigenvalue. We rewrite this equation as

$$A\vec{v} = \lambda I_2\vec{v} \qquad \text{so that} \qquad (A - \lambda I_2)\vec{v} = 0.$$

Thus the null space of

$$A - \lambda I_2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix},$$

contains more than the zero vector. It follows that

$$\det(A - \lambda I_2) = 0,$$

so that

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0.$$

Thus

$$0 = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4).$$

The roots of the characteristic polynomial are $\lambda = 2$ and $\lambda = 4$. If $\lambda = 2$ then the eigenspace of A is the nullspace of

$$A - 2I_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

If we apply Gaussian elimination we get

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

x is a basic variable and y is a free variable,

$$x + y = 0 \qquad \text{so that} \qquad x = -y.$$

$\vec{v}_1 = (1, -1)$ is an eigenvector with eigenvalue 2.
If $\lambda = 4$ then the eigenspace of A is the nullspace of

$$A - 2I_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

If we apply Gaussian elimination we get

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

x is a basic variable and y is a free variable,

$$x - y = 0 \qquad \text{so that} \qquad x = y.$$

$\vec{v}_2 = (1, 1)$ is an eigenvector with eigenvalue 4.
Note that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal. Let

$$\vec{u}_1 = \frac{1}{\sqrt{2}}(1, -1) \qquad \text{and} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}}(1, 1).$$

Then $\vec{u}_1$ and $\vec{u}_2$ are orthonormal. We have

$$A = PDP^{-1},$$

where P is the matrix whose columns are $\vec{u}_1$ and $\vec{u}_2$,

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Note that P is an orthogonal matrix, that is, the columns of P are orthonormal. So the inverse of P is the transpose of P. Therefore

$$A = PDP^{-1} = PDP^T.$$

We check:

$$\begin{aligned} PDP^T &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \end{aligned}$$

as expected.

Before we proceed, let's record the key property of symmetric matrices:

Lemma 28.3. *Let A be a symmetric matrix.*

If $\vec{v}$ and $\vec{w}$ are eigenvectors with distinct eigenvalues λ and μ then $\vec{v}$ and $\vec{w}$ are orthogonal.

Proof. Consider $(A\vec{v}) \cdot \vec{w}$. Note that

$$(A\vec{v}) \cdot \vec{w} = (A\vec{v})^T\vec{w} = \vec{v}^T A^T \vec{w} = \vec{v}^T A\vec{w} = \vec{v} \cdot A\vec{w}.$$

But

$$A\vec{v} \cdot \vec{w} = \lambda\vec{v} \cdot \vec{w} \qquad \text{and} \qquad \vec{v} \cdot A\vec{w} = \vec{v} \cdot \mu\vec{w} = \mu\vec{v} \cdot \vec{w}.$$

So

$$\lambda\vec{v} \cdot \vec{w} = \mu(\vec{v} \cdot \vec{w}).$$

As $\lambda \neq \mu$ we must $\vec{v} \cdot \vec{w} = 0$. But then $\vec{v}$ and $\vec{w}$ are orthogonal. □

Theorem 28.4. *Let A be a symmetric matrix.*

Then we can find a diagonal matrix A and an orthogonal matrix P such that

$$A = PDP^T.$$

In particular every symmetric matrix is diagonalisable.

Example 28.5. *Is*

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{pmatrix}$$

diagonalisable?

Yes, since it is symmetric. Let's diagonalise it. The characteristic polynomial is

$$\begin{aligned}\begin{vmatrix}1-\lambda & 1 & 3\\ 1 & 3-\lambda & 1\\ 3 & 1 & 1-\lambda\end{vmatrix} &= (1-\lambda)\begin{vmatrix}3-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix} - \begin{vmatrix}1 & 1\\ 3 & 1-\lambda\end{vmatrix} + 3\begin{vmatrix}1 & 3-\lambda\\ 3 & 1\end{vmatrix}\\ &= (1-\lambda)(3-\lambda)(1-\lambda) - (1-\lambda) - (1-\lambda) + 3 + 3 - 9(3-\lambda)\\ &= -\lambda^3 + 5\lambda^2 + 4\lambda - 20\\ &= -(\lambda-2)(\lambda+2)(\lambda-5).\end{aligned}$$

So the eigenvalues are 2, -2 and 5. We calculate the corresponding eigenvectors.

If $\lambda = 2$ we want the nullspace of $A - 2I_2$:

$$\begin{pmatrix}-1 & 1 & 3\\ 1 & 1 & 1\\ 3 & 1 & -1\end{pmatrix} \to \begin{pmatrix}1 & 1 & 1\\ -1 & 1 & 3\\ 3 & 1 & -1\end{pmatrix} \to \begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 4\\ 0 & -2 & -4\end{pmatrix} \to \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}.$$

The elimination is complete. x and y are basic variables, z is a free variable. Let's put $z = 1$:

$$y + 2 = 0 \qquad \text{so that} \qquad y = -2.$$

But then

$$x - 2 + 1 = 0 \qquad \text{so that} \qquad x = 1.$$

Thus $\vec{v}_1 = (1, -2, 1)$ is an eigenvector with eigenvalue $\lambda_1 = 2$.

If $\lambda = -2$ we want the nullspace of $A + 2I_2$:

$$\begin{pmatrix}3 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 3\end{pmatrix} \to \begin{pmatrix}1 & 5 & 1\\ 3 & 1 & 3\\ 3 & 1 & 3\end{pmatrix} \to \begin{pmatrix}1 & 5 & 1\\ 0 & -14 & 0\\ 0 & -14 & 0\end{pmatrix} \to \begin{pmatrix}1 & 5 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

The elimination is complete. x and y are basic variables, z is a free variable. Let's put $z = 1$:

$$y = 0$$

But then

$$x + 1 = 0 \qquad \text{so that} \qquad x = -1.$$

Thus $\vec{v}_2 = (-1, 0, 1)$ is an eigenvector with eigenvalue $\lambda_2 = -2$.

If $\lambda = 5$ we want the nullspace of $A - 5I_2$:

$$\begin{pmatrix}-4 & 1 & 3\\ 1 & -2 & 1\\ 3 & 1 & -4\end{pmatrix} \to \begin{pmatrix}1 & -2 & 1\\ -4 & 1 & 3\\ 3 & 1 & -4\end{pmatrix} \to \begin{pmatrix}1 & -2 & 1\\ 0 & -7 & 7\\ 0 & 7 & -7\end{pmatrix} \to \begin{pmatrix}1 & -2 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{pmatrix}.$$

The elimination is complete. x and y are basic variables, z is a free variable. Let's put $z = 1$:

$$y - 1 = 0 \qquad \text{so that} \qquad y = 1.$$

But then

$$x - 2 + 1 = 0 \qquad \text{so that} \qquad x = 1.$$

Thus $\vec{v}_1 = (1, 1, 1)$ is an eigenvector with eigenvalue $\lambda_1 = 5$.

Therefore

$$A = PDP^T$$

where

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \text{and} \qquad P = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & -\sqrt{3} & \sqrt{2} \\ -2 & 0 & \sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \end{pmatrix}$$