26. Gram-Schmidt

Gram-Schmidt is an algorithm that starts with any basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ for a linear space $W \subset \mathbb{R}^n$ and ends with an orthogonal basis $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$.

The idea is to construct $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k$ step by step.

At the first step we put $\vec{u}_1 = \vec{v}_1$. At the next step we adjust $\vec{v}_2$:

$$\vec{u}_2 = \vec{v}_2 - \alpha\vec{u}_1$$

so that it is orthogonal to $\vec{u}_1$. We want

$$(\vec{v}_2 - \alpha\vec{u}_1) \cdot \vec{u}_1 = 0 \qquad \text{so we choose} \qquad \alpha = \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}.$$

At the next step we adjust $\vec{v}_3$:

$$\vec{u}_3 = \vec{v}_3 - \beta\vec{u}_1 - \gamma\vec{u}_2,$$

so that is orthogonal to $\vec{u}_1$ and $\vec{u}_2$, and so on.

Example 26.1. *Find an orthogonal basis for the plane spanned by* $\vec{v}_1 = (3, 0, -1)$ *and* $\vec{v}_2 = (8, 5, -6)$.

We start with $\vec{u}_1 = (3, 0, -1)$. We adjust $\vec{v}_2$ so that it is orthogonal to $\vec{u}_1$:

$$\vec{u}_2 = \vec{v}_2 - \alpha\vec{u}_1 = (8, 5, -6) - \alpha(3, 0, -1).$$

We set

$$\alpha = \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{(8, 5, -6) \cdot (3, 0, -1)}{(3, 0, -1) \cdot (3, 0, -1)} = \frac{30}{10} = 3.$$

So

$$\vec{u}_2 = \vec{v}_2 - \alpha\vec{v}_1 = (8, 5, -6) - 3(3, 0, -1) = (-1, 5, -3).$$

Let's check:

$$\vec{u}_1 \cdot \vec{u}_2 = (3, 0, -1) \cdot (-1, 5, -3) = 0,$$

as expected.

Example 26.2. *Find an orthogonal basis for the column space of*

$$A = \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$$

Let $\vec{v}_1 = (3, 1, -1, 3)$, $\vec{v}_2 = (-5, 1, 5, -7)$ and $\vec{v}_3 = (1, 1, -2, 8)$.

We apply Gram-Schmidt $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

We start with $\vec{u}_1 = \vec{v}_1 = (3, 1, -1, 3)$. We adjust $\vec{v}_2$ so that it is orthogonal to $\vec{u}_1$:

$$\vec{u}_2 = \vec{v}_2 - \alpha\vec{u}_1 = (-5, 1, 5, -7) - \alpha(3, 1, -1, 3).$$

We want

$$\alpha = \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{(-5,1,5,-7)\cdot(3,1,-1,3)}{(3,1,-1,3)\cdot(3,1,-1,3)} = \frac{-40}{20} = -2.$$

So

$$\vec{u}_2 = (-5,1,5,-7) + 2(3,1,-1,3) = (1,3,3,-1).$$

Let's check:

$$\vec{u}_1 \cdot \vec{u}_2 = (3,1,-1,3)\cdot(1,3,3,-1) = 0$$

as expected.

Now we adjust $\vec{v}_3$ so that it is orthogonal to $\vec{u}_1$ and $\vec{u}_2$:

$$\vec{u}_3 = \vec{v}_3 - \beta\vec{u}_1 - \gamma\vec{u}_2 = (1,1,-2,8) - \beta(3,1,-1,3) - \gamma(1,3,3,-1).$$

We want

$$\beta = \frac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{(1,1,-2,8)\cdot(3,1,-1,3)}{(3,1,-1,3)\cdot(3,1,-1,3)} = \frac{30}{20} = \frac{3}{2}.$$

and

$$\gamma = \frac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{(1,1,-2,8)\cdot(1,3,3,-1)}{(1,3,3,-1)\cdot(1,3,3,-1)} = \frac{-10}{20} = -\frac{1}{2}.$$

So

$$\vec{u}_3 = \vec{v}_3 - \beta\vec{u}_1 - \gamma\vec{u}_2 = (1,1,-2,8) - \frac{3}{2}(3,1,-1,3) + \frac{1}{2}(1,3,3,-1) = (-3,1,1,3).$$

Let's check:

$$\vec{u}_1 \cdot \vec{u}_3 = (3,1,-1,3)\cdot(-3,1,1,3) = 0$$

and

$$\vec{u}_2 \cdot \vec{u}_3 = (1,3,3,-1)\cdot(-3,1,1,3) = 0$$

as expected.

Note that we can refine the orthogonal basis $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ to an orthonormal basis, simply by dividing through by the length, $\sqrt{20} = 2\sqrt{5}$.