22. 2nd midterm review

Let's suppose we start with an $m \times n$ matrix A and we consider the problem of trying to solve a system of linear equations:

$$A\vec{x} = \vec{b}.$$

There are two natural questions:

(1) For which $\vec{b} \in \mathbb{R}^m$ is there a solution?
(2) If for some $\vec{b}$ there is a solution $\vec{x} \in \mathbb{R}^n$, how many solutions can we find?

If $A\vec{x} = \vec{b}$ then $\vec{b}$ is a linear combination of the columns $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ of A:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{b}.$$

This is the answer to question #1.

Note that if $\vec{x}_p$ is a solution to $A\vec{x} = \vec{b}$ then the set of all solutions has the form $\vec{x}_p + \vec{x}_h$, where $\vec{x}_h$ is any solution to the homogeneous:

$$A\vec{x} = \vec{0}.$$

So if the equation $A\vec{x} = \vec{b}$ has one solution then it has as many solutions as the homogeneous. This is the answer to question #2.

Both the linear span and the solutions to the homogeneous are examples of linear subspaces:

$H \subset \mathbb{R}^n$ is a linear subspace if

(1) $\vec{0} \in H$,
(2) H is closed under addition, $\vec{u} \in H$ and $\vec{v} \in H$ implies $\vec{u}+\vec{v} \in H$,
(3) H is closed under scalar multiplication, $\vec{u} \in H$ and λ a scalar implies $\lambda\vec{u} \in H$.

The span of the columns is called the column space and the solutions to the homogeneous is called the nullspace.

The most basic question one can ask about a linear subspace is what is the dimension, the size of a basis. A basis is a set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ which are both independent and span.

Rank-nullity says that the rank of A, the dimension of the column space plus the nullity of A, the dimension of the nullspace, is n.

Example 22.1. *What is a basis for the column space, the row space, the nullspace and what is the rank and nullity of the following matrix:*

$$\begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix}?$$

We apply Gaussian elimination:

$$\to \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 14 & -35 & 42 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & 1 & -5/2 & 3 \\ 0 & 2 & -5 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There are pivots in the first and second column. The first and second columns of A are a basis for the column space:

$$\vec{c}_1 = (1, -1, 5) \qquad \text{and} \qquad \vec{c}_2 = (-4, 2, -6).$$

The rank is 2, the dimension of the column space.

There are pivots in the first and second row. The first and second row of the endproduct of elimination are a basis for the row space:

$$\vec{r}_1 = (1, -4, 9, -7) \qquad \text{and} \qquad \vec{r}_2 = (0, 1, -5/2, 3).$$

Note that the dimension of the row space is the same as the dimension of the column space. This is part of the statement of rank-nullity.

To find the nullspace we need to solve the homogeneous. We do this by back substitution. x and y are basic variables, z and w are free variables.

$$y - 5z/2 + 3w = 0 \qquad \text{so that} \qquad y = 5z/2 - 3w.$$

But then

$$x - 10z + 12w + 9z - 7w = 0 \qquad \text{so that} \qquad x = z - 5w.$$

The general solution is:

$$(x, y, z, w) = (z - 5w, 5z/2 - 3w, z, w) = z(1, 5/2, 1, 0) + w(-5, -3, 0, 1).$$

A basis is given by

$$\vec{n}_1 = (2, 5, 2, 0) \qquad \text{and} \qquad \vec{n}_2 = (-5, -2, 0, 1).$$

The nullity is 2. Note that the rank plus the nullity is $2 + 2 = 4$, as expected.

Example 22.2. *Are the vectors* $\vec{v}_1 = (-1, 3, 5, 4)$, $\vec{v}_2 = (2, 4, 2, 2)$, $\vec{v}_3 = (3, 3, 6, 4)$, $\vec{v}_4 = (0, 0, 6, 3)$ *a basis of* $\mathbb{R}^4$*?*

Let A be the matrix whose columns are the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_4$:

$$\begin{pmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 2 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{pmatrix}$$

Then $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_4$ are a basis if and only if A invertible. Indeed, $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_4$ are a basis if and only if

$$A\vec{x} = \vec{b}$$

has exactly one solution. This happens if and only if A is invertible.

To check whether or not A is invertible we could either apply Gaussian elimination or compute the determinant:

$$
\begin{aligned}
\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 2 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} &= \begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 2 & 0 & 6 \\ 4 & 2 & 1 & 3 \end{vmatrix} \\
&= \begin{vmatrix} -4 & -2 & 0 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 2 & 0 & 6 \\ 4 & 2 & 1 & 3 \end{vmatrix} \\
&= 2 \cdot 3 \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 2 & 0 & 2 \\ 4 & 2 & 1 & 1 \end{vmatrix} \\
&= 2 \cdot 3 \begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & 4 & 3 & 0 \\ 5 & 2 & 0 & 2 \\ -1 & 0 & 1 & -1 \end{vmatrix} \\
&= 2 \cdot 3 (2 \begin{vmatrix} 4 & 3 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 3 & 3 & 0 \\ 5 & 0 & 6 \\ -1 & 1 & -3 \end{vmatrix}) \\
&= -2 \cdot 3 (2 \cdot 2 \begin{vmatrix} 1 & 0 & 1 \\ 4 & 3 & 0 \\ 0 & 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} 1 & 1 & 0 \\ 5 & 0 & 2 \\ -1 & 1 & -1 \end{vmatrix}) \\
&= -2 \cdot 3 (2 \cdot 2 (\begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 4 & 0 \\ 0 & 1 \end{vmatrix}) + 3 (\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} - \begin{vmatrix} 5 & 2 \\ -1 & -1 \end{vmatrix})) \\
&= -2 \cdot 3 (2 \cdot 2 (-3 + 4) + 3(-2 + 3)) \\
&= 2 \cdot 3 (2 \cdot 2 + 3) \\
&= 42.
\end{aligned}
$$

Therefore $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ and $\vec{v}_4$ are a basis.