21. The dot product

Definition 21.1. *The* dot product *of two vectors* $\vec{u}$ *and* $\vec{v}$ *in* $\mathbb{R}^n$ *is the sum*

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example 21.2. *The dot product of* $\vec{u} = (1, 1)$ *and* $\vec{v} = (2, -1)$ *is*

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 + 1 \cdot -1 = 1.$$

The dot product of $\vec{u} = (1, 2, 3)$ *and* $\vec{v} = (2, -1, 1)$ *is*

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 + 2 \cdot -1 + 3 \cdot 1 = 3.$$

Note that when we compute the product of two matrices A and B in essence we are computing an array of dot products. In particular the dot product can be identified with the matrix product $\vec{u}^T \cdot \vec{v}$.

Definition 21.3. *The* length *of a vector* $\vec{v} \in \mathbb{R}^n$ *is the square root of the dot product of* $\vec{v}$ *with itself:*

$$\|\vec{v}\| = \sqrt{\vec{v}.\vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Note that

$$\|(x, y)\| = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$$

the usual formula for the length, using Pythagoras.

Example 21.4. *What is the length of the vector* $\vec{v} = (1, -2, 2)$*?*

$$\vec{v} \cdot \vec{v} = 1^2 + 2^2 + 2^2 = 9.$$

So the length is 3.

Note that the vector

$$\hat{u} = \frac{1}{3}\vec{v} = (1/3, -2/3, 2/3)$$

is a vector of unit length with the same direction as $\vec{v}$.

Definition 21.5. *Let* p *and* q *be two points in* $\mathbb{R}^n$.

The distance *between* P *and* Q *is the length of the vector*

$$\vec{v} = q - p.$$

Let $p = (1, 1, 1)$ and $q = (2, -1, 3)$. Then

$$\vec{v} = (2, -1, 3) - (1, 1, 1) = (1, -2, 1).$$

So the distance between p and q is 3, the length of $\vec{v}$.

Definition 21.6. *We say two vectors are* orthogonal *if* $\vec{u} \cdot \vec{v} = 0$.

The standard basis vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ of $\mathbb{R}^n$ are orthogonal.

Example 21.7. *Are the vectors $\vec{u} = (1, 1, -2)$ and $\vec{v} = (2, 0, 1)$ orthogonal?*

$$\vec{u} \cdot \vec{v} = (1, 1, -2) \cdot (2, 0, 1) = 2 + 0 - 2 = 0,$$

so that $\vec{u}$ and $\vec{v}$ are orthogonal.

Definition-Theorem 21.8. *Let $W \subset \mathbb{R}^n$ be a linear subspace. The* ***orthogonal complement*** *of W is*

$$W^{\perp} = \{\, v \in \mathbb{R}^n \,|\, v \cdot w = 0 \,\},$$

the set of all vectors which are orthogonal to every vector in W.

Then $W^{\perp}$ is a linear subspace of $\mathbb{R}^n$.

For example, suppose we start with a plane H in $\mathbb{R}^3$ through the origin. Then there is a line L in $\mathbb{R}^3$ through the origin which is the orthogonal complement of H:

$$L = H^{\perp}.$$

The line L is spanned by a vector which is orthogonal to every vector in H. Note that the relation between L and H is reciprocal, H is the orthogonal complement of L:

$$H = L^{\perp}.$$

Theorem 21.9. *Let A be an $m \times n$ matrix.*

The orthogonal complement of the row space of A is the null space of A and the orthogonal complement of the column space of A is the null space of A^T:

$$(\operatorname{Row} A)^{\perp} = \operatorname{Nul} A \qquad \textit{and} \qquad (\operatorname{Col} A)^{\perp} = \operatorname{Nul} A^T.$$

Proof. The rows of A correspond to equations. If a row is given by the vector

$$\vec{a} = (a_1, a_2, \dots, a_n)$$

then the corresponding equation is

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0.$$

$\vec{x}$ is in the null space if and only if it satisfies every equation.

But $\vec{x}$ satisfies the equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

if and only if the dot product $\vec{a} \cdot \vec{x} = 0$.

Thus $\vec{x}$ is in the null space if and only if it is in the orthogonal complement of the row space.

Now consider the matrix A^T. By what we just proved the null space of A^T is the orthogonal complement of the row space of A^T. But the row space of A^T is nothing but the column space of A. $\square$