20. Diagonalisation

Definition 20.1. *Let A and B be two square $n \times n$ matrices. We say that A and B are similar if there is an invertible square $n \times n$ matrix P such that $A = PBP^{-1}$.*

We say that A is diagonalisable if A is similar to a diagonal matrix D.

Suppose that

$$A = PBP^{-1}.$$

Then

$$\begin{aligned} A^2 &= A \cdot A \\ &= (PBP^{-1})(PBP^{-1}) \\ &= PB(P^{-1}P)BP^{-1} \\ &= PBBP^{-1} \\ &= PB^2P^{-1}. \end{aligned}$$

More generally we have:

Lemma 20.2. *Suppose that A and B are two $n \times n$ square matrices and that P is an invertible matrix such that*

$$A = PBP^{-1}.$$

Then

$$A^n = PB^nP^{-1}.$$

Proof. We prove this by induction on n. It is true for $n = 1$ by assumption. Suppose that

$$A^n = PB^nP^{-1},$$

for some $n > 0$. Then

$$\begin{aligned} A^{n+1} &= A \cdot A^n \\ &= (PBP^{-1})(PB^nP^{-1}) \\ &= PB(P^{-1}P)B^nP^{-1} \\ &= PBB^nP^{-1} \\ &= PB^{n+1}P^{-1} \end{aligned}$$

as required. Thus the result holds by induction on n. □

(20.2) gives us a practical way to compute the powers of a diagonalisable matrix A.

Theorem 20.3. *Let A be an $n \times n$ matrix.*

Then A is diagonalisable if and only if we can find a basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ of eigenvectors for $\mathbb{R}^n$. In this case,

$$A = PDP^{-1},$$

where P is the matrix whose columns are the eigenvectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ and D is the diagonal matrix whose diagonal entries are the corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

Proof. Suppose that $A = PDP^{-1}$, where the columns of P are $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ and D is a diagonal matrix with entries $\lambda_1, \lambda_2, \dots, \lambda_n$. We have

$$\begin{aligned} A\vec{v}_i &= (PDP^{-1})(P\vec{e}_i) \\ &= (PD)(P^{-1}P)\vec{e}_i \\ &= P(D\vec{e}_i) \\ &= P(\lambda_i\vec{e}_i) \\ &= \lambda_i(P\vec{e}_i) \\ &= \lambda_i\vec{v}_i. \end{aligned}$$

Therefore $\vec{v}_i$ is an eigenvector with eigenvalue λ_i. The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are a basis of $\mathbb{R}^n$ as P is invertible.

Now for the other direction. Suppose that $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are a basis of eigenvectors. Let P be the matrix whose columns are the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. Then P is an invertible matrix. Let $D = P^{-1}AP$. Then

$$\begin{aligned} D\vec{e}_i &= (P^{-1}AP)\vec{e}_i \\ &= P^{-1}A\vec{v}_i \\ &= P^{-1}\lambda_i\vec{v}_i \\ &= \lambda_i P^{-1}\vec{v}_i \\ &= \lambda_i\vec{e}_i. \end{aligned}$$

So D is the matrix whose ith row is the vector $\lambda_i\vec{e}_i$. But then D is a diagonal matrix with entries $\lambda_1, \lambda_2, \dots, \lambda_n$ on the main diagonal. We have

$$D = P^{-1}AP.$$

Multiplying both sides by P on the left, we get

$$PD = AP.$$

Finally multiplying both sides on the right by P^{-1} we get

$$A = PDP^{-1}. \qquad \square$$

Let's illustrate (20.3) by going back to the examples in §19.

Example 20.4. *Is it possible to diagonalise*

$$A = \begin{pmatrix} -8 & 5 \\ -10 & 7 \end{pmatrix}?$$

If the answer is yes, then diagonalise A.

We already saw that

$$\vec{v}_1 = (1, 2)$$

is an eigenvector with eigenvalue 2, and

$$\vec{v}_2 = (1, 1)$$

is an eigenvector with eigenvalue -3.

$\vec{v}_1$ and $\vec{v}_2$ are independent (either by inspection or because $2 \neq -3$). Thus A is diagonalisable.

Let

$$P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

the matrix whose columns are the eigenvectors. Then

$$P^{-1} = -1 \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Let

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

the diagonal matrix whose entries on the diagonal are the eigenvalues. Let's compute PDP^{-1}:

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 2 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} -8 & 5 \\ -10 & 7 \end{pmatrix},$$

as expected.

Now we can compute any power of A easily:

$$A^n = PD^nP^{-1}.$$

We compute

$$\begin{aligned} \begin{pmatrix} -8 & 5 \\ -10 & 7 \end{pmatrix}^n &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & (-3)^n \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -2^n & 2^n \\ 2(-3)^n & -(-3)^n \end{pmatrix} \\ &= \begin{pmatrix} -2^n + 2(-3)^n & 2^n - (-3)^n \\ -2^{n+1} + 2(-3)^n & 2^{n+1} - (-3)^n \end{pmatrix}. \end{aligned}$$

Example 20.5. *Is it possible to diagonalise*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}?$$

If the answer is yes, then diagonalise A.

We already saw that

$$\vec{v}_1 = (-1, -6, 13)$$

is an eigenvector with eigenvalue 0,

$$\vec{v}_2 = (-1, 2, 1)$$

is an eigenvector with eigenvalue -4, and

$$\vec{v}_3 = (-2, -3, 2)$$

is an eigenvector with eigenvalue 3.

$\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are independent, since their eigenvalues 0, -4 and 3 are distinct. Therefore they are a basis and so A is diagonalisable.

Let

$$P = \begin{pmatrix} -1 & -1 & -2 \\ -6 & 2 & -3 \\ 13 & 1 & 2 \end{pmatrix}$$

the matrix whose columns are the eigenvectors. Then, with the aid of a computer,

$$P^{-1} = \frac{1}{84} \begin{pmatrix} 7 & 0 & 7 \\ -27 & 24 & 9 \\ -32 & -12 & -8 \end{pmatrix}.$$

Let

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

the diagonal matrix whose entries on the diagonal are the eigenvalues. Now we can compute any power of A easily:

$$A^n = PD^nP^{-1}.$$

We compute

$$\frac{1}{84} \begin{pmatrix} -1 & -1 & -2 \\ -6 & 2 & -3 \\ 13 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-4)^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} \begin{pmatrix} 7 & 0 & 7 \\ -27 & 24 & 9 \\ -32 & -12 & -8 \end{pmatrix}.$$

Example 20.6. *Is it possible to diagonalise*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

If the answer is yes, then diagonalise A.

We compute the eigenvalues of A:

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2.$$

So the only eigenvalue is $\lambda = 1$. We want to compute the null space of $A - I_2$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

y is a basic variable and x is a free variable. $y = 0$. Thus $\vec{e}_1$ is an eigenvector with eigenvalue 1. A is not diagonalisable, we cannot find a basis of eigenvalues.

Example 20.7. *Is it possible to diagonalise*

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}.$$

If the answer is yes, then diagonalise A.

The characteristic polynomial is:

$$\begin{aligned} \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} &= -(1+\lambda)\begin{vmatrix} -\lambda & -3 \\ 0 & -1-\lambda \end{vmatrix} + \begin{vmatrix} 3 & -\lambda \\ 1 & 0 \end{vmatrix} \\ &= -(1+\lambda)^2\lambda + \lambda \\ &= -\lambda^2(\lambda+2). \end{aligned}$$

Thus the eigenvalues are 0 and -2.

We want to calculate the nullspace of A. We apply Gaussian elimination:

$$\begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

x is a basic variable, y and z are free variables.

$$x - z = 0 \qquad \text{so that} \qquad x = z.$$

The general solution is

$$(x, y, z) = (z, y, z) = y(0, 1, 0) + z(1, 0, 1).$$

A basis for the nullspace is given by $(0,1,0)$ and $(1,0,1)$.

$$\vec{v}_1 = (0,1,0) \qquad \text{and} \qquad \vec{v}_2 = (1,0,1)$$

are independent eigenvectors with eigenvalue 0.

We want to calculate the nullspace of $A + 2I_3$.

$$A + 2I_3 = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix}.$$

We apply Gaussian elimination:

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}.$$

x and y are basic variables, z is a free variable.

$$y - 3z = 0 \qquad \text{so that} \qquad y = 3z.$$

Therefore

$$x + z = 0 \qquad \text{so that} \qquad x = -z.$$

The general solution is

$$(x, y, z) = (-z, 3z, z) = z(-1, 3, 1).$$

$$\vec{v}_3 = (-1, 3, 1)$$

is an eigenvector with eigenvalue -2. The vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are independent, thus A is diagonalisable.

Let

$$P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

the matrix whose columns are the eigenvectors. Then, with the aid of a computer,

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 3 & 1 & -3 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Let

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

the diagonal matrix whose entries on the diagonal are the eigenvalues. Now we can compute any power of A easily:

$$A^n = PD^nP^{-1}.$$

We compute

$$\frac{1}{2}\begin{pmatrix}0 & 1 & -1\\ 1 & 0 & 3\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & (-2)^n\end{pmatrix}\begin{pmatrix}3 & 1 & -3\\ 1 & 0 & 1\\ -1 & 0 & 1\end{pmatrix}.$$