2. Echelon form

Given a system of linear equations, the first step of Gaussian elimination is to write down the augmented matrix. Corresponding to the three basic operations on linear equations, there are three basic row operations

- add a multiple of one row to another,
- multiply one row by a scalar,
- swap two rows.

We call two matrices row equivalent if we can get from one to the other by these basic operations. The key point is that two augmented matrices which are row equivalent have the same solutions.

As a proof of concept, let us look at a slightly more complicated example. Suppose that we start with a system of four equations in four unknowns,

$$\begin{aligned} x + 3y - 2z - w &= -1 \\ -6x - 15y + 9z + 9w &= 9 \\ -x \qquad - z + 4w &= 5 \\ 4x + 10y - 5z - 2w &= -3. \end{aligned}$$

We first replace this by the augmented matrix,

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & -1 \\ -6 & -15 & 9 & 9 & 9 \\ -1 & 0 & -1 & 4 & 5 \\ 4 & 10 & -5 & -2 & -3 \end{array}\right).$$

The first step is to use the one in the first row, first column to eliminate the -6, -1 and 4 from the first column. To do this we add 6, 1, -4 times the first row to the second, third and fourth row, to get

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & -1 \\ 0 & 3 & -3 & 3 & 3 \\ 0 & 3 & -3 & 3 & 4 \\ 0 & -2 & 3 & 2 & 1 \end{array}\right).$$

The next step is to multiply the second row by $1/3$ to get a one in the second row, second column,

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 3 & 4 \\ 0 & -2 & 3 & 2 & 1 \end{array}\right).$$

Now we eliminate the 3 and the -2 in the second column. To do this we add -3 and 2 times the second row to the third and fourth row,

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 & 3 \end{array}\right).$$

The next step is to get a one in the third row, third column. We cannot do this by rescaling the third row. We can do it by swapping the third and fourth rows,

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Consider what happens when we try to solve the resulting linear equations by back substitution. The last equation reads

$$0x + 0y + 0z + 0w = 1.$$

There are no values for x, y, z and w which work. The original system has no solutions; it is inconsistent. Now suppose we start with the system

$$\begin{aligned} x + 3y - 2z - w &= -1 \\ -6x - 15y + 9z + 9w &= 9 \\ -x \qquad - z + 4w &= 4 \\ 4x + 10y - 5z - 2w &= -5. \end{aligned}$$

The only thing we have changed is the third number on the right from 5 to $4 = 5 - 1$. If we follow the same steps as before, we get down to the same matrix, except that the last entry is $0 = 1 - 1$ (remember at some point we swapped two rows)

$$\left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The last equation

$$0x + 0y + 0z + 0w = 0,$$

places no restriction on x, y, z and w. The previous equation reads,

$$z + 4w = 3.$$

Using this equation, we can solve for z in terms of w, to get $z = 3-4w$. We can use this value for z in the second equation to determine y in terms of w,

$$y - (3 - 4w) + w = 1,$$

to get $y = 4 - 5w$. Finally, we can use this value for y and z in the first equation to solve for x,

$$x + 3(4 - 5w) - 2(3 - 4w) - w = -1,$$

to get

$$x = -7 + 8w.$$

We get the family of solutions,

$$(x, y, z, w) = (-7 + 8w, 4 - 5w, 3 - 4w, w).$$

We can represent the solutions in a slightly different way,

$$(x, y, z, w) = (-7, 4, 3, 0) + w(8, -5, -4, 1).$$

Note that no matter the value of w, we get a solution for the original system of linear equations. We call w a free variable. The other variables are called basic variables. We can express the basic variables in terms of the free variables.

For example if we pick $w = 0$, we get the solution

$$(x, y, z, w) = (-7, 4, 3, 0),$$

but if we choose the value $w = 1$, we get the solution

$$(x, y, z, w) = (3, -1, -1, 1).$$

In particular this system has infinitely many solutions.

The reason for this is because the original system of equations are not independent. In fact even the first three equations are not independent. We can think of this as being the same thing as the first three rows of the augmented matrix are not independent and the row of zeroes in the coefficient matrix was in the third row. Trying small numbers (or tracking the Gaussian elimination), one can check that the third row is equal to the second row plus five times the first row. It is then automatic that any solution to the first equation and the second equations is a solution to the third equation.

In fact there is only one dependence between the equations (or rows) the dependence between the first three equations. Note that in this case, one dependence gives one free variable. Four equations, four unknowns, we expect one solution. In fact the equations have one dependence which gives one free variable.

It is interesting to also note that what went wrong with the previous set of linear equations is closely related. In this case the rows of the coefficient matrix are not independent.

We say that a matrix is in row echelon form if it is the endproduct of Gaussian elimination. Given a particular matrix, it is easy to say whether the matrix is in row echelon form or not. It is surprisingly hard to actually give a definition of row echelon form:

Definition 2.1. *Let A be a matrix. We say that A is in (row) echelon form if A satisfies the following properties:*

- *The first (reading left to right) non-zero entry in every row is a "1". Such entries are called pivots.*
- *Every row which contains a pivot occurs before (reading top to bottom) every row which does not contain a pivot.*
- *Pivots in earlier rows (again, reading top to bottom) come in earlier columns (again, reading left to right).*

$$\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 & -4 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are in row echelon form.

$$\begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 & -4 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

are not in row echelon form.

In terms of Gaussian elimination, in the first matrix we should switch the rows, in the second matrix we should multiply the second row by $1/2$ and in the third matrix we should switch the second and third rows.

In terms of (2.1), the first matrix breaks the third rule, the second matrix breaks the first rule and the third matrix breaks the second rule.

Theorem 2.2. *A linear system of equations is inconsistent if and only if the last column of the row echelon form of the matrix contains a pivot.*

Proof. If the last entry of some row is a pivot then this row looks like

$$(0, 0, 0, 0, \dots, 0, 1).$$

The corresponding equation reads

$$0x_1 + 0x_2 + 0x_3 + \cdots + 0x_n = 1 \qquad \text{that is} \qquad 0 = 1,$$

which is impossible (or inconsistent). Otherwise if there are no pivots in the last column then one can solve the equations by back-substitution. The basic variables correspond to the pivots and the free variables are the remaining variables. □