19. The characteristic equation

If $\vec{v}$ is an eigenvector with eigenvalue λ then

$$A\vec{v} = \lambda\vec{v}.$$

As observed above if we rewrite the RHS as $\lambda I_n \vec{v}$ then

$$A\vec{v} = \lambda I_n \vec{v} \qquad \text{so that} \qquad (A - \lambda I_n)\vec{v} = \vec{0}.$$

It follows that if λ is an eigenvalue of A then the null space of $A - \lambda I_n$ is non-trivial, so that $A - \lambda I_n$ is not invertible. In this case the determinant of $A - \lambda I_n$ is zero. We can use the determinant to find the eigenvalues of A.

Example 19.1. *What are the eigenvalues and eigenvectors of*

$$A = \begin{pmatrix} -8 & 5 \\ -10 & 7 \end{pmatrix}?$$

We want to find the vectors $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v},$$

for some scalar λ. We rewrite this as

$$(A - \lambda I_2)\vec{v} = \vec{0}.$$

We want to know when the matrix

$$A - \lambda I_2 = \begin{pmatrix} -8-\lambda & 5 \\ -10 & 7-\lambda \end{pmatrix}$$

is not invertible. This is the same as to say that the determinant is zero. So we want those scalars λ such that

$$\begin{vmatrix} -8-\lambda & 5 \\ -10 & 7-\lambda \end{vmatrix} = (-8-\lambda)(7-\lambda) + 50 = 0.$$

Rearranging we get

$$\lambda^2 + \lambda - 6 = 0.$$

$\lambda^2 + \lambda - 6$ is called the characteristic polynomial and $\lambda^2 + \lambda - 6 = 0$ is called the characteristic equation.

The roots of the characteristic polynomial are

$$\lambda = 2 \qquad \text{and} \qquad \lambda = -3.$$

The eigenvalues are $\lambda = 2$ and $\lambda = -3$.

What are the corresponding eigenvectors?

If $\lambda = 2$ then we want to calculate the nullspace of

$$A - 2I_2 = \begin{pmatrix} -10 & 5 \\ -10 & 5 \end{pmatrix}.$$

If we apply Gaussian elimination we get

$$\begin{pmatrix} 1 & -1/2 \\ 0 & 0 \end{pmatrix}.$$

x is a basic variable and y is a free variable. If $y = 1$ then $x = 1/2$. So

$$(1/2, 1).$$

If we multiply by 2 we still get an eigenvector:

$$\vec{v}_1 = (1, 2)$$

is an eigenvector with eigenvalue 2. Let's check:

$$\begin{pmatrix} -8 & 5 \\ -10 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

as expected.

If $\lambda = -3$ then we want to calculate the nullspace of

$$A + 3I_2 = \begin{pmatrix} -5 & 5 \\ -10 & 10 \end{pmatrix}.$$

If we apply Gaussian elimination we get

$$\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

x is a basic variable and y is a free variable. If $y = 1$ then $x = 1$. So

$$\vec{v}_2 = (1, 1)$$

is an eigenvector with eigenvalue -3. Let's check:

$$\begin{pmatrix} -8 & 5 \\ -10 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

as expected.

Definition-Theorem 19.2. *Let A be an $n \times n$ matrix A.*

The equation

$$\det(A - \lambda I_n) = 0$$

is called the characteristic equation.

The roots of the characteristic polynomial are the eigenvalues of A.

Proof. If $\vec{v}$ is an eigenvector with eigenvalue λ then

$$A\vec{v} = \lambda\vec{v}.$$

Rearranging we get

$$(A - \lambda I_n)\vec{v} = \vec{0}.$$

So λ is an eigenvalue if and only if A is not invertible if and only if $\det(A - \lambda I_n) = 0$. Thus λ is a root of the characteristic polynomial. □

Example 19.3. *What are the eigenvalues and eigenvectors of*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}?$$

As usual we are looking for vectors $\vec{v}$ and scalars λ such that

$$A\vec{v} = \lambda\vec{v}.$$

Rearranging we get

$$(A - \lambda I_3)\vec{v} = \vec{0}.$$

We are looking for roots of the characteristic equation. The characteristic polynomial is

$$\begin{aligned}
\begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix} &= \begin{vmatrix} 1-\lambda & 6 & -1 \\ 2 & -1-\lambda & -2 \\ 1 & 0 & -1-\lambda \end{vmatrix} \\
&= -\begin{vmatrix} 1 & 0 & -1-\lambda \\ 2 & -1-\lambda & -2 \\ 1-\lambda & 6 & -1 \end{vmatrix} \\
&= -1\begin{vmatrix} -1-\lambda & -2 \\ 6 & -1 \end{vmatrix} + (1+\lambda)\begin{vmatrix} 2 & -1-\lambda \\ 1-\lambda & 6 \end{vmatrix} \\
&= -1(1+\lambda+12) + (1+\lambda)(12 + (1-\lambda)(1+\lambda)) \\
&= -13 - \lambda + 12 + 12\lambda + (1+\lambda)(1-\lambda^2) \\
&= 12\lambda - \lambda^2 - \lambda^3 \\
&= -\lambda(\lambda^2 + \lambda - 12) \\
&= -\lambda(\lambda - 3)(\lambda + 4).
\end{aligned}$$

The characteristic equation is

$$\lambda(\lambda - 3)(\lambda + 4) = 0.$$

So the eigenvalues are $\lambda = 0$, $\lambda = 3$ and $\lambda = -4$.

Let's calculate the corresponding eigenvectors. If $\lambda = 0$ we want to calculate the nullspace of A. Let's apply Gaussian elimination:

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -13 & -6 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 6/13 \\ 0 & 0 & 0 \end{pmatrix}.$$

The elimination is complete. x and y are basic variables and z is a free variable. If $z = 1$ then using the second equation we get

$$y + 6/13 = 0 \qquad \text{so that} \qquad y = -6/13.$$

But then

$$x - 12/13 + 1 = 0 \qquad \text{so that} \qquad x = -1/13.$$

We get

$$(x, y, z) = (-1/13, -6/13, 1).$$

Multiplying through by 13 to get

$$\vec{v}_1 = (-1, -6, 13).$$

Let's check:

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$$

as expected.

If $\lambda = -4$ we want to calculate the nullspace of $A + 4I_3$:

$$A + 4I_3 = \begin{pmatrix} -2 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix}.$$

Let's apply Gaussian elimination:

$$\begin{pmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 5 & 2 & 1 \\ 6 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & -8 & 16 \\ 0 & -9 & 18 \end{pmatrix}$$

so that

$$\rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & -9 & 18 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The elimination is complete. x and y are basic variables and z is a free variable. If $z = 1$ then using the second equation we get

$$y - 2 = 0 \qquad \text{so that} \qquad y = 2.$$

But then

$$x + 4 - 3 = 0 \qquad \text{so that} \qquad x = -1.$$

So

$$\vec{v}_2 = (-1, 2, 1)$$

is an eigenvector with eigenvalue -4. Let's check:

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix} = -4 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

as expected.

If $\lambda = 3$ we want to calculate the nullspace of $A - 3I_3$:

$$A - 3I_3 = \begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix}.$$

Let's apply Gaussian elimination:

$$\begin{pmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1/2 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1/2 \\ 0 & 2 & 3 \\ 0 & -3 & -9/2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The elimination is complete. x and y are basic variables and z is a free variable. If $z = 2$ then using the second equation we get

$$y + 3 = 0 \qquad \text{so that} \qquad y = -3.$$

But then

$$x + 3 - 1 = 0 \qquad \text{so that} \qquad x = -2.$$

Thus

$$\vec{v}_3 = (-2, -3, 2)$$

is an eigenvector with eigenvalue 3. Let's check:

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ -9 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} -6 \\ -9 \\ 6 \end{pmatrix}$$

as expected.