16. The determinant of a matrix

Suppose that we have an $n \times n$ matrix A. Then we expect a unique solution to any equation

$$A\vec{x} = \vec{b}.$$

But sometimes this does not work. Is there a quick way to tell if there is a unique solution? Yes and no. In fact there is a single number, called the determinant, which answers this question. Unfortunately it is not easy to compute this number.

Let A be a $n \times n$ square matrix. Then, up to sign, the determinant is the product of all the numbers which appear on the main diagonal in Gaussian elimination (just before we make these entries one). For example, consider

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

We apply Gaussian elimination. Multiplying the first row by -3 and adding it to the second row we get,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

So the determinant is -2. Similarly if the matrix is

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

the determinant is $1 \cdot 4 \cdot 6 = 24$.

Given a matrix A we either write $\det A$ or

$$\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -2$$

for the determinant.

The most important feature of the determinant are its basic properties:

1. $\det I_n = 1$.
2. If one swaps two rows then the determinant changes sign.
3. If one adds a multiple of one row to another row then the determinant is unchanged.
4. If we multiply a row by a scalar λ then the determinant is also multiplied by λ.
5. $\det A^T = \det A$.
6. $\det AB = \det A \det B$.

From these basic properties, we get some interesting consequences:

(i) By (5) any rule which applies to rows also applies to columns.
(ii) From either (2) or (3) one sees that if a row is repeated then the determinant is zero.
(iii) From property (1) and (6), we see that

$$\det A^{-1} = \frac{1}{\det A}.$$

In particular if the determinant is zero then A cannot be invertible.

Note that (2-4) cover the elementary row operations. This explains why one can compute the determinant using Gaussian elimination.

16.1. **Cofactors.** There is a seemingly simple way to compute determinants. For a start, the determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Consider

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix}.$$

The idea is to expand about the top row. For the first entry, cross out the first row and column and take 1 times the corresponding 2×2 determinant. For the second entry take -2 time the determinant of the 2×2 matrix obtained by crossing out the first row and second column. Thus

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix} = 1 \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix},$$

which comes out to

$$1(4-0) - 2(8-3) + 1(0-1) = 4 - 10 - 1 = -7.$$

In practive though this way of computing things is very inefficient. However if we use some of the rules above, we can make the whole computation much easier:

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 4 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix},$$

since if we swap two rows the sign changes. Now if we expand about the top row, the computation is much easier:

$$\begin{vmatrix} 1 & 0 & 4 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 1 \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + 4 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

which comes out to

$$1(6-1)+4(1-4)=5-12=-7,$$

the same as before. Here is a more complicated example

$$\begin{vmatrix} 1 & 2 & 3 & -5 \\ 2 & 4 & 7 & 11 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & -1 & 7 \end{vmatrix} = - \begin{vmatrix} 0 & 0 & 1 & 0 \\ 2 & 4 & 7 & 11 \\ 1 & 2 & 3 & -5 \\ 2 & 1 & -1 & 7 \end{vmatrix},$$

swapping the first and third rows. Expanding about the top row gives

$$- \begin{vmatrix} 2 & 4 & 11 \\ 1 & 2 & -5 \\ 2 & 1 & 7 \end{vmatrix}$$

Swapping the first and second rows gives

$$\begin{vmatrix} 1 & 2 & -5 \\ 2 & 4 & 11 \\ 2 & 1 & 7 \end{vmatrix}$$

Now use the fact that the second row is almost twice the first row. Subtracting twice the first row from the second row gives

$$\begin{vmatrix} 1 & 2 & -5 \\ 0 & 0 & 21 \\ 2 & 1 & 7 \end{vmatrix}$$

Switching rows gives,

$$- \begin{vmatrix} 0 & 0 & 21 \\ 1 & 2 & -5 \\ 2 & 1 & 7 \end{vmatrix},$$

and expanding about the top row we get

$$-21 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -21(1-4) = 63.$$

Clearly we would not have wanted to calculate this determinant by using the method of cofactors.