14. The rank and nullity

We already saw that to get a basis for the column space of a matrix, just compute the echelon form and then the number of pivots is the dimension of the column space; the pivot columns of A are a basis for the column space.

Definition 14.1. *Let A be an $m \times n$ matrix. The rank of A is the dimension of the column space.*

Theorem 14.2. *The rank of a matrix is the number of pivots.*

What about the nullspace of a matrix A and the row space?

Definition 14.3. *Let A be an $m \times n$ matrix. The nullity of A is the dimension of the nullspace.*

Let's first understand the row space. The basic idea is that elementary row operations won't change the row space, so we can compute the row space by applying Gaussian elimination.

Let's start with

$$A = \begin{pmatrix} 1 & 9 & -4 \\ -1 & -9 & 2 \\ 5 & 45 & -20 \end{pmatrix}.$$

We apply Gaussian elimination:

$$\begin{pmatrix} 1 & 9 & -4 \\ -1 & -9 & 2 \\ 5 & 45 & -20 \end{pmatrix} \to \begin{pmatrix} 1 & 9 & -4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 9 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rows with the pivots are independent and they span the row space, since the other rows are zero. The row space has dimension two and a basis is given by

$$\vec{r}_1 = (1, 9, -4) \qquad \text{and} \qquad \vec{r}_2 = (0, 0, 1).$$

The column space has dimension two as well. The columns corresponding to the pivots are a basis of the column space:

$$\vec{c}_1 = (1, -1, 5) \qquad \text{and} \qquad \vec{c}_2 = (-4, 2, -20).$$

What about the nullity and the nullspace? We calculate this using Gaussian elimination. If the variables are x, y, z then x and z are the basic variables and y is a free variable.

We solve for x and z in terms of y:

$$z = 0.$$

and so

$$x + 9y = 0 \qquad \text{so that} \qquad x = -9y.$$

It follows that

$$(x, y, z) = (-9y, y, 0) = y(-9, 1, 0).$$

The nullspace is the span of $(-9, 1, 0)$. The nullity is one.

Theorem 14.4 (Rank-nullity)**.** *Let A be an $m \times n$ matrix.*

The dimension of the column space and the dimension of the row space are the same, both equal to the rank of A. The rank of A plus the nullity of A is equal to n.

Proof. Apply Gaussian elimination to get the echelon form of A. The dimension of the column space of A is the number of pivot columns and the dimension of the row space is the number of pivots, and so the dimension of the column space and the row is space is the same.

The nullity is the number of free variables. There are n variables and they are divided into basic variables and free variables. The basic variables correspond to the pivots, so the number of basic variables is equal to the rank. So the rank plus the nullity is n. □

It is interesting to think of (14.4) from a different viewpoint. Let's suppose that we have a linear function

$$f : \mathbb{R}^5 \longrightarrow \mathbb{R}^3.$$

f corresponds to a 3×5 matrix A. The column space corresponds to the image of f.

There are various possibilities. We'd expect f to be onto, so that the image is the whole of $\mathbb{R}^3$ and the column space is the whole of $\mathbb{R}^3$. In this case the rank is 3. Rank-nullity says that the nullspace has dimension 2.

But perhaps the image of f is a plane in $\mathbb{R}^3$. In this case the column space is a plane, which has dimension 2. To compensate the nullspace has dimension 3.

Or perhaps the image of f is a line in $\mathbb{R}^3$. In this case the column space is a line, which has dimension 1. To compensate the nullspace has dimension 4.

Or perhaps the image of f is a point in $\mathbb{R}^3$, the origin. This is the most extremee case. The column space is a point, which has dimension 0. To compensate the nullspace has dimension 5, everything is sent to zero, so that the nullspace is the whole of $\mathbb{R}^5$.

It is also interesting to think about this in terms of equations. The 3×5 matrix corresponds to three equations in five unknowns. We'd expect to be always able to solve this system of equations. If we can always solve the equations then there are three pivots and two free variables. The general solution is described using two parameters and so the nullspace is two dimensional.

But perhaps we can't always solve this equation. The dimension of the nullspace is equal to the difference $n - m$ plus the failure of the linear equations to impose independent conditions.