13. Dimension of a linear subspace

The most interesting thing is to figure out the dimension of a linear subspace. The definition is just the same, with H replacing $\mathbb{R}^m$:

Definition 13.1. *The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in H \subset \mathbb{R}^m$ are a basis of H if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are both independent and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span H.*
The dimension of H is n, the size of a basis.

Example 13.2. *Let's take a line in $\mathbb{R}^3$, for example the line*

$$(x, y, z) = \lambda(1, 1, 1).$$

This is the span of $\vec{v}_1 = (1, 1, 1)$. In particular it is a linear subspace. The vector $\vec{v}_1$ is independent. Thus $\{\vec{v}_1\}$ is a basis and so the line has dimension one.

Suppose we take $\vec{v}_1 = (1, 1, 1)$ and $\vec{v}_2 = (2, 2, 2)$. These vectors together span the line but they are not independent;

$$-2\vec{v}_1 + \vec{v}_2 = \vec{0}.$$

They are not a basis.

Example 13.3. *Let's take a plane in $\mathbb{R}^3$, for example the plane*

$$x + 2y + 3z = 0.$$

This is a linear subspace, the nullspace of

$$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

We want to find a basis of the nullspace. x is a basic variable and y and z are free variables. We solve for x in terms of y and z:

$$x + 2y + 3z = 0 \qquad \text{so that} \qquad z = -2y - 3z.$$

Parametrically,

$$(x, y, z) = (-2y - 3z, y, z) = y(-2, 1, 0) + z(-3, 0, 1).$$

Clearly the vectors $\vec{v}_1 = (-2, 1, 0)$ and $\vec{v}_2 = (-3, 0, 1)$ span the solution set H. $\vec{v}_1$ and $\vec{v}_2$ are not parallel, so that they are independent. Thus $\{\vec{v}_1, \vec{v}_2\}$ is a basis and the plane H has dimension two.

Theorem 13.4. *If $H \subset \mathbb{R}^m$ is a linear subspace then the dimension of H is at most m.*

Proof. Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is a basis of H. Then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are independent vectors in $\mathbb{R}^m$. It follows that $p \leq m$. But p is the dimension of H. $\qquad \square$

How do we find a basis for the column space of a matrix? The column space of a matrix is spanned by the columns of A. So the columns $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ span the column space. The only problem is that they might not be independent. We need to discard some of the columns. Which columns are redundant?

Lemma 13.5. *If $\vec{v}_p$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{p-1}$ then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{p-1}$ spans the same space as $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.*

Proof. Clearly, anything which is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{p-1}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Suppose that $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$. We have to show that $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{p-1}$. By assumption $\vec{v}_p$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{p-1}$ so we may find scalars $a_1, a_2, \ldots, a_{p-1}$ such that

$$\vec{v}_p = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_{p-1}\vec{v}_{p-1}.$$

As $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ we may find scalars $x_1, x_2, \ldots, x_p$ such that

$$\vec{v} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_p \vec{v}_p.$$

Then

$$\vec{v} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_p \vec{v}_p$$
$$= x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_p(a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_{p-1}\vec{v}_{p-1})$$
$$= (x_1 + x_p a_1)\vec{v}_1 + (x_2 + x_p a_2)\vec{v}_2 + \cdots + (x_{p-1} + x_p a_{p-1})\vec{v}_{p-1}.$$

Thus $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{p-1}$. $\qquad \square$

Example 13.6. *Find a basis for the space spanned by the vectors*

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 2 \\ -2 \\ -8 \\ 0 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 2 \\ -1 \\ 10 \\ 3 \end{pmatrix} \quad \vec{v}_5 = \begin{pmatrix} 3 \\ -1 \\ -6 \\ 9 \end{pmatrix}.$$

This is the same as the column space of the matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ -2 & 2 & -8 & 10 & -6 \\ 3 & 3 & 0 & 3 & 9 \end{pmatrix}.$$

The trick is to apply Gaussian elimination.

$$\begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ -2 & 2 & -8 & 10 & -6 \\ 3 & 3 & 0 & 3 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ 0 & 2 & -4 & 14 & 0 \\ 0 & 3 & -6 & -3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 16 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1/8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The key point is that the elementary row operations preserve the relations between the columns. Let's focus on the columns of A which have pivots, that is, the first, second, fourth and fifth columns. If we make a matrix with just those columns,

$$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1/8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then we see that the first, second, fourth and fifth columns are independent.

Now look at the third column. It is linear combination of the first and second columns. The relevant part of the matrix is the 2×3 uppper left corner (a priori we should focus on the left 4×3 matrix, but we can forget the last two rows, since they are all zeroes):

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \end{pmatrix}$$

Clearly the third column is twice the first column plus minus twice the second column. Thus

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \quad \vec{v}_4 = \begin{pmatrix} 2 \\ -1 \\ 10 \\ 3 \end{pmatrix} \quad \vec{v}_5 = \begin{pmatrix} 3 \\ -1 \\ -6 \\ 9 \end{pmatrix}$$

are a basis.

$$2\vec{v}_1 - 2\vec{v}_2 = (1, 0, -2, 3) + (0, 1, 2, 3) = (2, 2, -8, 0) = \vec{v}_3,$$

as expected. The span of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ and $\vec{v}_5$ has dimension four.

In general the dimension of the column space of a matrix A is the number of pivots; a basis is given by the pivot columns and the other column vectors are a linear combination of those.