12. Basis and dimension

Recall two definitions:

Definition 12.1. *The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$ are (linearly) dependent if there are scalars $x_1, x_2, \dots, x_m$, not all zero, such that*

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}.$$

We say that the vectors are (linearly) independent if they are not dependent.

Linear independence places a restriction on the number n of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. If the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$ are independent then $n \leq m$. You cannot have too many independent vectors.

At the other extreme we have:

Definition 12.2. *We say that the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$ spans $\mathbb{R}^m$ if every vector $\vec{b} \in \mathbb{R}^m$ is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.*

Vectors which span places a restriction on the number n of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. If the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$ span $\mathbb{R}^m$ then $n \geq m$. You cannot have too few vectors which span.

Definition 12.3. *The vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$ are a basis of $\mathbb{R}^m$ if $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are both independent and $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span $\mathbb{R}^m$.*

The dimension of $\mathbb{R}^m$ is n, the size of a basis.

Since we already observed that if the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are independent then $n \leq m$ and if the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span $\mathbb{R}^m$ then $n \geq m$. Therefore we must have $n = m$. Thus $\mathbb{R}^m$ has dimension m.

In fact we have:

Theorem 12.4. *Let A be an $n \times n$ matrix.*

The columns of A are a basis of $\mathbb{R}^n$ if and only if A is invertible.

Example 12.5. *Let I_n be the identity matrix. Then I_n is invertible. The columns of A are*

$$\vec{e}_1 = (1, 0, \dots, 0), \qquad \vec{e}_2 = (0, 1, \dots, 0) \qquad \text{and} \qquad \vec{e}_n = (0, 0, \dots, 1)$$

a basis of $\mathbb{R}^n$, called the standard basis.

Example 12.6. *Consider the vectors $\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (1, -1)$.*

We make a matrix with these columns:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The determinant is

$$ad - bc = 1 \cdot -1 - 1 \cdot 1 = -2 \neq 0.$$

This matrix is invertible. The vectors $\vec{v}_1 = (1, 1)$ and $\vec{v}_2 = (1, -1)$ are a basis of $\mathbb{R}^2$.

In fact it is not hard to see this directly. $\vec{v}_1$ and $\vec{v}_2$ are not parallel, so they are independent. Two independent vectors in $\mathbb{R}^2$ always span. One can see this both algebraically and geometrically. Algebraically, if two vectors in the plane are independent then the homogeneous equation $A\vec{x} = \vec{0}$ has only one solution, the obvious solution $\vec{x} = (0, 0)$. In this case A must have two pivots and so there are no rows of zeroes. But then the equation $A\vec{x} = \vec{b}$ is always consistent and the two vectors $\vec{v}_1$ and $\vec{v}_2$ span $\mathbb{R}^2$.