10. Vector subspaces

The solution to a homogeneous equation $A\vec{x} = \vec{0}$ in $\mathbb{R}^3$ is one of

- The origin.
- A line through the origin.
- A plane through the origin.
- The whole of $\mathbb{R}^3$.

These are all examples of linear subspaces.

Definition 10.1. *Let H be a subset of $\mathbb{R}^n$.*
H is called a linear subspace if

(1) $\vec{0} \in H$.
(2) H is closed under addition: If $\vec{u}$ and $\vec{v} \in H$ then $\vec{u} + \vec{v} \in H$.
(3) H is closed under scalar multiplication: If $\vec{u}$ and λ is a scalar then $\lambda\vec{u} \in H$.

Geometrically H is closed under scalar multiplication if and only if H is a union of lines through the origin. H is then closed under addition if and only if it contains every plane containing ever pair of lines.

Example 10.2. *Let $H = \{\vec{0}\}$. Then H is a linear subspace. Indeed, $\vec{0} \in H$. $\vec{0} + \vec{0} = \vec{0} \in H$. Similarly $\lambda\vec{0} = \vec{0}$.*

Example 10.3. *Let $H = \mathbb{R}^n$. Then H is a linear subspace. Indeed, $\vec{0} \in H$. H is obviously closed under addition and scalar multiplication.*

Now consider lines in $\mathbb{R}^3$.

Example 10.4. *Let H be the x-axis. Then H is a linear subspace. Indeed, $\vec{0} \in H$. If $\vec{u}$ and $\vec{v}$ belong to H then $\vec{u}$ and $\vec{v}$ are multiples of $(1, 0, 0)$ and the sum is a multiple of $(1, 0, 0)$. Similarly if λ is a scalar then $\lambda\vec{u}$ is a multiple of $(1, 0, 0)$.*

Example 10.5. *Let H be a line in $\mathbb{R}^3$ through the origin. Then H is a linear subspace. Indeed, $\vec{0} \in H$. The elements of H are all multiples of the same vector $\vec{w}$. If $\vec{u}$ and $\vec{v}$ are in H then $\vec{u}$ and $\vec{v}$ are multiples of $\vec{w}$. The sum is a multiple of $\vec{w}$. Thus $\vec{u} + \vec{v} \in H$. Similarly if λ is a scalar then $\lambda\vec{u}$ is a multiple of $\vec{w}$.*

It is interesting to see what happens when we don't have a linear subspace:

Example 10.6. *Let*

$$H = \{ (x, y) \mid y = x^2 \} \subset \mathbb{R}^2,$$

a parabola, the graph of $y = x^2$. This does contain the origin. Consider the vector $(1,1) \in H$ and the vector $(2,4) \in H$. The sum is

$$(1,1) + (2,4) = (3,5) \notin H.$$

Similarly $(1,1) \in H$ but $2(1,1) = (2,2) \notin H$.

H is neither closed under addition nor under scalar multiplication. H is not a linear subspace.

Theorem 10.7. *Let $v_1, v_2, \dots, v_p$ be vectors in $\mathbb{R}^n$ and let*

$$H = \operatorname{span}\{\, \vec{u} \,|\, \vec{u} \text{ is a linear combination of } v_1, v_2, \dots, v_p \,\}$$

be the span.

Then H is a linear subspace of $\mathbb{R}^n$.

Proof. $\vec{0} \in H$, since $\vec{0}$ is a linear combination of $v_1, v_2, \dots, v_p$ (use zero weights). If $\vec{u}$ and $\vec{v}$ belong to H then $\vec{u}$ and $\vec{v}$ are linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$. Suppose that

$$\vec{u} = x_1\vec{v}_1 + x_2\vec{x}_2 + \dots + x_p\vec{x}_p \qquad \text{and} \qquad \vec{v} = y_1\vec{v}_1 + y_2\vec{x}_2 + \dots + y_p\vec{x}_p,$$

for scalars $x_1, x_2, \dots, x_p$ and $y_1, y_2, \dots, y_p$. Then

$$\vec{u} + \vec{v} = (x_1 + y_1)\vec{v}_1 + (x_2 + y_2)\vec{x}_2 + \dots (x_p + y_p)\vec{x}_p$$

is a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ so that $\vec{u} + \vec{v} \in H$. So H is closed under addition. If λ is a scalar then

$$\lambda\vec{u} = (\lambda x_1)\vec{v}_1 + (\lambda x_2)\vec{x}_2 + \dots (\lambda x_p)\vec{x}_p.$$

So H is closed under scalar multiplication. Thus H is a linear subspace. □

If we are given a matrix A the span of the columns $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ of A is called the column space of A, $\operatorname{col}(A)$.

There is one other way to produce lots of linear subspaces:

Definition-Theorem 10.8. *Let A be a matrix. The solutions to the homogeneous equation $A\vec{x} = \vec{0}$ is a linear subspace H, called the nullspace of A, $\operatorname{null}(A)$.*

Proof. $\vec{0} \in H = \operatorname{null}(A)$. If $\vec{u}$ and $\vec{v} \in H = \operatorname{null}(A)$ then

$$A\vec{u} = \vec{0} \qquad \text{and} \qquad A\vec{v} = \vec{0}.$$

But then

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}.$$

Thus $\vec{u} + \vec{v} \in \operatorname{null}(A)$ and $\operatorname{null}(A)$ is closed under addition. Similarly if λ is a scalar then

$$A(\lambda\vec{u}) = \lambda(A\vec{u}) = \lambda\vec{0} = \vec{0}.$$

Thus $\lambda\vec{u} \in \text{null}(A)$ and $\text{null}(A)$ is closed under scalar multiplication.

Thus $\text{null}(A)$ is a linear subspace. □

Example 10.9. *Let H be the plane $2x - 4y + 7z = 0$ in $\mathbb{R}^3$.*

Then H is a linear subspace of $\mathbb{R}^3$. Indeed, let

$$A = \begin{pmatrix} 2 & -4 & 7 \end{pmatrix}.$$

Then $H = \text{null}(A)$ is the nullspace of A.

Example 10.10. *Let H be the first quadrant in $\mathbb{R}^2$,*

$$H = \{\, (x, y) \,|\, x \geq 0 \textit{ and } y \geq 0 \,\}.$$

Then H doesn't look like a linear subspace. Let's check that it isn't. $0 \in H$ and in fact H is closed under addition. If $\vec{u} = (a, b)$ and $\vec{v} = (c, d)$ then

$$\vec{u} + \vec{v} = (a + b, c + d).$$

$a + b \geq 0$ and $c + d \geq 0$ so that $\vec{u} + \vec{v} \in H$.

Suppose we take $\lambda = 2$. Then

$$\lambda\vec{u} = 2(a, b) = (2a, 2b).$$

But suppose that we take $\vec{u} = (1, 0)$ and $\lambda = -1$. Then

$$\lambda\vec{u} = -1(1, 0) = (-1, 0) \notin H.$$

So H is not closed under scalar multiplication. H is not a linear subspace.

Consider polynomials of degree at most 2 in the variable t. For example

$$f(t) = 3 - 4t + 6t^2 \qquad \text{or} \qquad g(t) = 3 - 5t.$$

The general polynomial of degree at most two looks like

$$p(t) = a_0 + a_1 + a_2t^2.$$

Note that we can add polynomials,

$$f(t) + g(t) = (3 - 4t + 6t^2) + (3 - 5t) = 6 - 9t + 6t^2$$

and multiply them by a scalar

$$3f(t) = 3(3 - 4t + 6t^2) = 6 - 12t + 18t^2.$$

There is even a zero polynomial.

$$q(t) = 0.$$

All of the basic rules of algebra which apply to vectors apply to polynomials. For example if we add the zero polynomial to another polynomial nothing happens.

P_n denotes the set of polynomials of degree at most n in the variable t. We think of P_n as being an abstract vector space, in which case we will call the elements of P_n vectors.

Definition 10.11. *Let $f\colon V \longrightarrow W$ be a function between vector spaces. We say that f is linear if*

(1) It is additive: $f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w})$ for all vectors $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$.

(2) $f(\lambda\vec{v}) = \lambda f(\vec{v})$, for all scalars λ and vectors $\vec{v} \in \mathbb{R}^n$.

Example 10.12. *Let*

$$f\colon P_n \longrightarrow P_{n-1} \qquad \text{given by} \qquad f(p(t)) = \frac{dp(t)}{dt}$$

by the function which associates to a polynomial of degree n the derivative.

The fact that f is linear follows from basic rules of differentiation:

$$\frac{d(p(t) + q(t))}{dt} = \frac{dp(t)}{dt} + \frac{dq(t)}{dt} \qquad \text{and} \qquad \frac{d}{dt}(\lambda p(t)) = \lambda\frac{dp(t)}{dt}.$$

The derivative of a sum is the sum of the derivatives; the derivative of a scalar multiple is the scalar multiples of the derivative.