1. Systems of linear equations

$$2x - 3y = 4 \qquad -x + 7y + 3z = 2 \qquad \text{and} \qquad 2x_1 - 4x_2 + 5x_3 - 7x_4 = 3$$

are all examples of linear equations. By contrast

$$x^2 - 5x + 6 = 0 \qquad 2xy + \cos(z) = 3 \qquad \text{and} \qquad e^x + 2y - z = 6$$

are not linear equations. x^2 is the problem in the first equation, both xy and $\cos(z)$ are the problem in the second equation and e^x is the dissident term in the last equation.

The solutions to the first linear equation is a line in $\mathbb{R}^2$, the solutions to the second linear equation is a plane in $\mathbb{R}^3$ and the solutions to the third linear equation is a 3-plane in 4-space (whatever that means).

We are interested in solving systems of linear equations, for example:

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 2, \end{aligned}$$

or

$$\begin{aligned} -x + 7y + 3z &= 2 \\ 2x - y + z &= 1 \\ \sqrt{2}x - 3y + 6z &= -3. \end{aligned}$$

Geometrically the first pair of linear equations represents the locus of points where two lines meet. By the same token the second triple of linear equations represents the locus of points in space where all three planes coincide.

There are three different qualitative possibilities for a configuration of two lines in the plane. The two lines might intersect in a point, the two lines might be parallel or the two lines might be the same line. The number of solutions to a pair of linear equations in two variables is therefore either

- zero, or
- one, or
- infinite.

No matter the number of equations, nor the number of variables, this is always the case.

We say that a linear system is inconsistent if there are no solutions and otherwise we also say that the system is consistent.

In applications we will be interested in solving large systems of equations, involving lots of variables and lots of equations. It is therefore important to use the most efficient method to solve the system.

Suppose we are given a system of equations. There are three basic operations on the system which won't change the set of solutions. We could

- add a multiple of one equation to another,
- multiply one equation by a scalar,
- swap two equations.

In fact it is obvious that the last operation won't affect the solutions, clear that multiplying an equation by a scalar won't change the solutions, and after a little bit of thought one can see that the first operation won't affect the solutions as well.

These are the only operations we need to find the solutions to any system of linear equations. Here is an example

$$\begin{aligned} x + y + z &= 1 \\ 2x + y + z &= 2 \\ -3x - 2y + z &= 0. \end{aligned}$$

Let's get rid of the first $2x$. Multiply the first equation by -2 and add it to the second equation

$$\begin{aligned} x + y + z &= 1 \\ -y - z &= 0 \\ -3x - 2y + z &= 0. \end{aligned}$$

Now let's eliminate $-3x$. Multiply the first equation by 3 and add it to the third equation

$$\begin{aligned} x + y + z &= 1 \\ -y - z &= 0 \\ y + 4z &= 3. \end{aligned}$$

Now we multiply the second equation by -1 to get

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 0 \\ y + 4z &= 3. \end{aligned}$$

To eliminate the last y, we multiply the second equation by -1 and add it to the third equation

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 0 \\ 3z &= 3. \end{aligned}$$

Finally we multiply the last equation by 1/3 to get

$$\begin{aligned} x + y + z &= 1 \\ y + z &= 0 \\ z &= 1. \end{aligned}$$

The elimination is complete. We solve this system of equations by back substitution. The last equation tell us $z = 1$. If we substitute $z = 1$ back into the second equation we get

$$y + 1 = 0 \qquad \text{so that} \qquad y = -1.$$

If we substitute $z = 1$ and $y = -1$ into the first equation we get

$$x - 1 + 1 = 1 \qquad \text{so that} \qquad x = 1.$$

The unique solution to the original system of linear equations is $x = 1$, $y = -1$ and $z = 1$. Notice that this is what we expect. Three equations, three unknowns, we expect one solution. Three planes in space, we expect them to intersect in a single point.

This algorithm is called Gaussian elimination. Gauss is widely recognised as one of the best mathematicians. There is only one algorithm due to Gauss and this algorithm accounts for nearly all of the computation done on computers.

To improve efficiency, it is useful to recognise we are carrying around some extra baggage. We only care about the coefficients, which form a 3×3 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ -3 & -2 & 1 \end{pmatrix}.$$

called the coefficient matrix and the values on the right, which we can combine with the coefficient matrix to form the augmented matrix

$$B = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ -3 & -2 & 1 & 0 \end{array}\right) = \begin{pmatrix} A & | & b \end{pmatrix}$$

where

$$b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

is a 3×1 matrix.

Now we run the same algorithm on the augmented matrix. We take the first row, multiply by -2 and add it to the second row to eliminate the 2 in the second row. Similarly we take the first row, multiply by 3 and add it to the third row to eliminate the 2 in the third row. We are left with

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 4 & 3 \end{array}\right).$$

Now we multiply the second row by -1

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 4 & 3 \end{array}\right).$$

Now we eliminate the 1 in the third row by adding -1 times the second row.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 3 \end{array}\right).$$

Finally we multiply the last row by $1/3$ to get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

As before, we can solve the resulting equations by back-substitution. In this case it makes sense to write the solution as $(x, y, z) = (1, -1, 1)$.