FINAL EXAM
MATH 20F, UCSD, AUTUMN 14

You have three hours.

Problem	Points	Score
1	25	
2	15	
3	15	
4	15	
5	15	
6	10	
7	10	
8	15	
9	10	
10	20	
11	10	
12	10	
13	10	
14	10	
15	10	
Total	200	

There are 15 problems, and the total number of points is 200. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

1. (25pts) (i) Show that the matrix

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix} \quad \text{is row equivalent to} \quad \begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

To get full credit for this problem, you **must** show your steps and explain what row operations you are using at each stage.

Solution: We swap the first and third rows; we multiply the first row by 1/3:

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix} \to \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

Then we multiply the first row by -3 and add it to the third row; we multiply the second row by 1/2; we multiply the second row by -3 and add it to the third row:

$$\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

(ii) Find the general solution to the linear equations in parametric form:

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15. \end{aligned}$$

Solution: By part (i) we can use

$$\begin{pmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

to solve the system by back substitution:
x_1, x_2 and x_5 are basic variables, x_3 and x_4 are free variables.
The last equation reads $x_5 = 4$. The second equation reads

$$x_2 - 2x_3 + 2x_4 + 4 = -3 \qquad \text{so that} \qquad x_2 = 2x_3 - 2x_4 - 7.$$

The first equation reads

$$x_1 - 3(2x_3 - 2x_4 - 7) + 4x_3 - 3x_4 + 8 = 5 \qquad \text{so that} \qquad x_1 = 2x_3 - 3x_4 - 24.$$

The general solution is

$$\begin{aligned} (x_1, x_2, x_3, x_4, x_5) &= (2x_3 - 3x_4 - 24, 2x_3 - 2x_4 - 7, x_3, x_4, 4) \\ &= (-24, -7, 0, 0, 4) + x_3(2, 2, 1, 0, 0) + x_4(-3, -2, 0, 1, 0). \end{aligned}$$

2. (15pts) Let

$$A = \begin{pmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 3 & -9 & 12 & -9 & 6 \end{pmatrix}$$

(i) Find a basis for the nullspace of A. What is the nullity of A?

Solution: We already saw in 1 (ii) that a basis for the null space is $(2, 2, 1, 0, 0)$ and $(-3, -2, 0, 1, 0)$. The nullity is 2.

(ii) Find a basis for the column space of A. What is the rank of A?

Solution: There are pivots in the first, second and fifth columns. $(0, 3, 3)$, $(3, -7, -9)$ and $(4, 8, 6)$ is a basis for the column space. The rank is 3.

(iii) Find a basis for the row space of A.

Solution: $(1, -3, 4, -3, 2)$, $(0, 1, -2, 2, 1)$, $(0, 0, 0, 0, 1)$ are a basis for the row space.

3. (15pts) For which values of h are the following vectors

$$\vec{v}_1 = (1, 1, 1) \qquad \vec{v}_2 = (1, 2, -1) \qquad \text{and} \qquad \vec{v}_3 = (1, h, -3)$$

a basis of $\mathbb{R}^3$?

Solution: Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & h \\ 1 & -1 & -3 \end{pmatrix}$$

be the matrix whose columns are the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. The vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are a basis if and only if A is invertible.

We apply Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & h \\ 1 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 0 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & h-1 \\ 0 & 0 & 2h-6 \end{pmatrix}$$

A is invertible if and only if $2h - 6 \neq 0$, that is, $h \neq 3$.

So the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are a basis if and only if $h \neq 3$

4. (15pts) (i) Let f be the linear function

$f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $(x, y, z) \longrightarrow (3x-2y+z, x+y+z, 2x+y-2z)$.

Find a matrix A such that $f(\vec{x}) = A\vec{x}$.

Solution:

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

(ii) Let g be the linear function

$g\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by $(x, y, z) \longrightarrow (x + y + z, 2x - 3y + z)$.

Find a matrix B such that $g(\vec{x}) = B\vec{x}$.

Solution:

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \end{pmatrix}.$$

(iii) Let $g \circ f$ be the composition of f and g. Find a matrix C such that $(g \circ f)(\vec{x}) = C\vec{x}$.

Solution:

$$C = BA = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 5 & -6 & -3 \end{pmatrix}.$$

5. (15pts) Let A be a matrix which is row equivalent to

$$U = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) Is the equation $A\vec{x} = \vec{b}$ consistent for every $\vec{b} \in \mathbb{R}^3$?

Solution: No.
Since there is a row of zeroes after row reduction of A we can pick a vector $\vec{b}$ so that there is a pivot in the last colum of the augmented matrix.

(ii) Suppose that $\vec{x} = (1, -1, 2, 0)$ is a solution to $A\vec{x} = \vec{b}$, where $\vec{b} = (1, 2, 3)$. What is the general solution to $A\vec{x} = \vec{b}$?

Solution: We solve the homogeneous by back substitution. a and c are basic variables, b and d are free.

$$c - 3d = 0 \qquad \text{so that} \qquad c = 3d.$$

Then

$$a + b - 2d = 0 \qquad \text{so that} \qquad a = -b + 2d.$$

The general solution is

$$(a, b, c, d) = (1, -1, 2, 0) + (-b+2d, b, 3d, d) = (1, -1, 2, 0) + b(-1, 1, 0, 0) + d(2, 0, 3, 1).$$

(iii) Find a basis for the row space of A.

Solution: $(1, 1, 0, -2)$ and $(0, 0, 1, -3)$.

6. (10pts) Is there a linear transformation $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $f(1,1) = (0,0)$, $f(1,-2) = (1,1)$ and $f(0,1) = (1,-1)$? If so, give an example of such an f; if not, explain why not.

Solution:
No.
Let A be the matrix associated to f. Since $f(1,1) = (0,0)$, A has a non-trivial null space. So the nullity of A is at least one. The vectors $(1,1)$ and $(1,-1)$ are in the image of f. These are independent, so the column space of A contains a plane. Thus the rank of f is at least two. But the rank plus the nullity is two.

7. (10pts) Let P_2 be the vector space of all polynomials of degree at most 2,

$$P_2 = \{\, a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R} \,\}.$$

Write $5t^2 - 2t - 3$ as a linear combination of $1 - t$ and $1 - t^2$.

Solution: Suppose that $5t^2 - 2t - 3 = a(1-t) + b(1-t^2)$. Then

$$(a+b) - at - bt^2 = 5t^2 - 2t - 3.$$

Comparing coefficients we must have:

$$a + b = -3 \qquad a = 2 \qquad \text{and} \qquad b = -5.$$

So $a = 2$ and $b = -5$. Thus

$$5t^2 - 2t - 3 = 2(1-t) - 5(1-t^2).$$

8. (15pts) What is the determinant of

$$A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 1 & 3 & 4 & 0 \end{pmatrix}?$$

Is A invertible?

Solution:

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 1 & 3 & 4 & 0 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 3 \\ -1 & 1 & 1 & 4 \\ 1 & 2 & 3 & 0 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 0 & 3 \\ 1 & 1 & 4 \\ 2 & 3 & 0 \end{vmatrix} - \begin{vmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} \\
&= -\begin{vmatrix} 1 & 4 \\ 3 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} \\
&= -\begin{vmatrix} 1 & 4 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix} \\
&= 12 + 1 + 4 \\
&= 17.
\end{aligned}$$

A is invertible as the determinant is non-zero.

9. (10pts) Find the inverse of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Solution: We apply Gauss-Jordan elimination to the super augmented matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

so that

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right).$$

The inverse matrix is

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

10. (20pts) (i) Check that the vectors $\vec{v}_1 = (1, 1, -1)$, $\vec{v}_2 = (0, 1, 1)$ and $\vec{v}_3 = (2, -1, 1)$ are an orthogonal basis of $\mathbb{R}^3$.

Solution: $\vec{v}_1 \cdot \vec{v}_2 = (1, 1, -1) \cdot (0, 1, 1) = 0$, $\vec{v}_1 \cdot \vec{v}_3 = (1, 1, -1) \cdot (2, -1, 1) = 0$, and $\vec{v}_2 \cdot \vec{v}_3 = (0, 1, 1) \cdot (2, -1, 1) = 0$. They are orthogonal and so they are independent. Hence they are a basis.

(ii) Let $\vec{v} = (5, 2, -2)$. Compute $\vec{v}_1 \cdot \vec{v}$, $\vec{v}_2 \cdot \vec{v}$ and $\vec{v}_3 \cdot \vec{v}$.

Solution: $\vec{v}_1 \cdot \vec{v} = (5, 2, -2) \cdot (1, 1, -1) = 9$, $\vec{v}_2 \cdot \vec{v} = (5, 2, -2) \cdot (0, 1, 1) = 0$ and $\vec{v}_3 \cdot \vec{v} = (5, 2, -2) \cdot (2, -1, 1) = 6$.

(iii) Compute $\vec{v}_1 \cdot \vec{v}_1$, $\vec{v}_2 \cdot \vec{v}_2$ and $\vec{v}_3 \cdot \vec{v}_3$.

Solution: $\vec{v}_1 \cdot \vec{v}_1 = (1, 1, -1) \cdot (1, 1, -1) = 3$, $\vec{v}_2 \cdot \vec{v}_2 = (0, 1, 1) \cdot (0, 1, 1) = 2$ and $\vec{v}_3 \cdot \vec{v}_3 = (2, -1, 1) \cdot (2, -1, 1) = 6$.

(iv) Write $\vec{v}$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

Solution:

$$\vec{v} = 3\vec{v}_1 + \vec{v}_3.$$

11. (10pts) Let W be the span of $\vec{v}_1 = (1,0,1,1)$, $\vec{v}_2 = (0,-1,1,1)$ and $\vec{v}_3 = (2,1,-2,3)$. Find an orthogonal basis of W.

Solution: We apply Gram-Schmidt:

$$\vec{u}_1 = \vec{v}_1.$$

Then

$$\vec{u}_2 = \vec{v}_2 - \alpha \vec{u}_1 \qquad \text{where} \qquad (\vec{v}_2 - \alpha \vec{u}_1) \cdot \vec{u}_1 = 0.$$

Therefore

$$\alpha = \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{2}{3}.$$

Thus

$$\vec{u}_2 = (0,-1,1,1) - \frac{2}{3}(1,0,1,1) = \frac{1}{3}(-2,-3,1,1).$$

Let's replace $\vec{u}_2$ by $(-2,-3,1,1)$. This is still orthogonal to $\vec{u}_1$. Then

$$\vec{u}_3 = \vec{v}_3 - \beta \vec{u}_1 - \gamma \vec{u}_2 \qquad \text{where} \qquad (\vec{v}_3 - \beta \vec{u}_1 - \gamma \vec{u}_2) \cdot \vec{u}_i = 0.$$

Therefore

$$\beta = \frac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{3}{3} = 1 \qquad \text{and} \qquad \gamma = \frac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{-4-3-2+3}{15} = -\frac{2}{5}.$$

Thus

$$\vec{u}_3 = (2,1,-2,3) - (1,0,1,1) + \frac{2}{5}(-2,-3,1,1) = \frac{1}{5}(1,-1,-13,12).$$

Hence

$$(1,0,1,1), \qquad (-2,-3,1,1) \qquad \text{and} \qquad (1,-1,-13,12),$$

is an orthogonal basis.

12. (10pts) Find the least squares solution to $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}.$$

Solution: We have to solve $(A^T A)\vec{x} = A^T\vec{b}$.

$$A^T = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

So

$$A^T A = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad A^T\vec{b} = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

To solve $A^T A\vec{x} = A^T\vec{b}$ we apply Gaussian elimination:

$$\left(\begin{array}{cc|c} 6 & 2 & 13 \\ 2 & 3 & 9 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/3 & 13/6 \\ 2 & 3 & 9 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/3 & 13/6 \\ 0 & 7/3 & 14/3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/3 & 13/6 \\ 0 & 1 & 2 \end{array}\right)$$

The elimination is complete. We solve by back substitution. $y = 2$ and $x + 2/3 = 13/6$ so that $x = 3/2$. The least squares solution is $(3/2, 2)$.

13. (10pts) Show that $\vec{v}_1 = (1, 0, -1)$ and $\vec{v}_2 = (2, -1, -1)$ are eigenvectors of

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

What are the eigenvalues?
Solution:

$$A\vec{v}_1 = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2\vec{v}_1$$

and

$$A\vec{v}_2 = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \vec{v}_2$$

Therefore $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors with eigenvalues 2 and 1.

14. (10pts) Let

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Given that

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 16 \\ 2 \\ 6 \end{pmatrix},$$

find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$ (there is no need to find P^{-1}).

Solution: The given vectors are eigenvectors with eigenvalues 1, -1 and 2. The eigenvalues are different so that the eigenvectors are automatically independent.
Let

$$P = \begin{pmatrix} 1 & 1 & 8 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

the matrix P whose columns are the eigenvectors and the diagonal matrix D whose entries are the eigenvalues. Then $A = PDP^{-1}$.

15. (10pts) Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(i) Find a basis for the eigenspace with eigenvalue -1.

Solution: We want the null space of $A + I_3$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

x is a basic variable, y and z are free variables. $x + y + z = 0$, so that $x = -y - z$ and the general solution is

$$(x, y, z) = (-y - z, y, z) = y(-1, 1, 0) + z(-1, 0, 1).$$

The eigenspace with eigenvalue -1 is the span of $(-1, 1, 0)$ and $(-1, 0, 1)$.

(ii) Does A have other eigenvalues? If so, identify them; if not, explain why not.

Solution: Yes. A is symmetric and there must be other eigenvectors. Consider expanding $\det(A - \lambda I_3)$ as a polynomial in λ. The constant term is the value of the determinant when $\lambda = 0$:

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 2.$$

But minus the constant term is the product of the eigenvalues. Thus the products of the eigenvalues is -2. As the product of two of them is 1 the third eigenvalue is -2.