MODEL ANSWERS TO HWK #7

1. Let U be the free abelian semigroup generated by $v_1, v_2, \dots, v_m$ (so that U is abstractly isomorphic to $\mathbb{N}^m$). Define a semigroup homomorphism $U \longrightarrow S_\sigma$ by sending v_i to u_i. This is surjective and the kernel is generated by relations of the form

$$\sum a_i v_i - \sum b_i v_i,$$

where

$$\sum a_i u_i = \sum b_i u_i$$

in S_σ. The group algebra A_σ is generated by $x_i = \chi^{u_i}$. Define a ring homomorphism

$$K[x_1, x_2, \dots, x_n] \longrightarrow A_\sigma,$$

by sending x_i to χ^{u_i}. Then the kernel certainly contains relations of the form

$$x_1^{a_1} x_2^{a_2} \dots x_m^{a_m} - x_1^{b_1} x_2^{b_2} \dots x_m^{b_m},$$

where

$$\sum a_i u_i = \sum b_i u_i.$$

If we quotient out by these relations, then we get a vector space Q with one dimensional eigenspaces indexed by $u \in S_\sigma$. As A_σ has the same property (the corresponding eigenspaces are spanned by the monomials χ^u) the induced linear map $Q \longrightarrow A_\sigma$ is an isomorphism. So the given relations actually generate the kernel.

2. 5.1. (a) Note that there are perfect pairings

$$\mathcal{E} \times \check{\mathcal{E}} \longrightarrow \mathcal{O}_X.$$

and

$$\check{\mathcal{E}} \times (\check{\mathcal{E}})\check{} \longrightarrow \mathcal{O}_X.$$

Thus $\mathcal{E} \simeq (\check{\mathcal{E}})\check{}$.

(b) Let $U \subset X$ be an open subset. There is a natural $\mathcal{O}_X(U)$-module homomorphism

$$\check{\mathcal{E}}(U) \underset{\mathcal{O}_X(U)}{\otimes} \mathcal{F}(U) \longrightarrow \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{E}|_U, \mathcal{F}_U).$$

This defines a homomorphism from the presheaf to the sheaf, and from there a morphism between sheaves.

This map is an isomorphism if and only if it is an isomorphism on stalks. So we may assume that $\mathcal{E}$ is free. Since both sides commute

with the direct sum, we may assume that $\mathcal{E} = \mathcal{O}_X$, in which case the result is clear, since both sides are $\mathcal{F}$.
(c) If M, N and P are three A-modules, then there is a natural isomorphism

$$\operatorname{Hom}_A(M \underset{A}{\otimes} N, P) \simeq \operatorname{Hom}_A(M, \operatorname{Hom}_A(N, P)).$$

(d)
5.2. (a) X has two open subsets, $U_0 = X = \{x_0, x_1\} = \operatorname{Spec} R$ and $U_1 = \{x_1\} = \operatorname{Spec} K$, which are both affine. Let $M = H^0(U_0, \mathcal{F})$ and $L = H^0(U_1, \mathcal{F})$. Then M is an R-module and L is a K-module. On the other hand, L is also the stalk at the generic point of an irreducible scheme, so that L is a field. Restricting sections from X to U_1 defines a homomorphism

$$\rho\colon M \underset{R}{\otimes} K \longrightarrow L$$

Conversely, a sheaf is nothing more than the data of M, L and the morphism ρ.
(b) If $\mathcal{F} = \tilde{M}$ then it is easy to see that ρ is an isomorphism. On the other hand, given M, there is exactly one quasi-coherent sheaf $\mathcal{F}$ such that $M = H^0(U_0, \mathcal{F})$. Thus $\mathcal{F}$ is a quasi-coherent sheaf if and only if ρ is an isomorphism.
5.3. Define a ring homomorphism

$$\rho\colon \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}) \longrightarrow \operatorname{Hom}_A(M, \Gamma(X, \mathcal{F})),$$

by taking global sections. Define a ring homomorphism

$$\sigma\colon \operatorname{Hom}_A(M, \Gamma(X, \mathcal{F})) \longrightarrow \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}),$$

as follows.
Suppose that $f \in A$ and $\phi\colon M \longrightarrow \Gamma(X, \mathcal{F})$. Then

$$(\tilde{M})(U_f) = M_f.$$

Further, $\mathcal{F}(U_f)$ is an A_f-module, so that ϕ induces a natural homomorphism

$$M_f \longrightarrow \mathcal{F}(U_f).$$

Composing gives us a homomorphism

$$\sigma(\phi)(U_f)\colon \tilde{M}(U_f) \longrightarrow \mathcal{F}(U_f).$$

As U_f are a base for the topology, we can glue these maps to get

$$\sigma(\phi)\colon \tilde{M} \longrightarrow \mathcal{F}.$$

This defines σ, and it is easy to see that this is the inverse to ρ.
5.4. We may as well assume that $X = \operatorname{Spec} A$. Suppose that $\mathcal{F}$ is quasi-coherent. Then there is an A-module M such that $\mathcal{F} = \tilde{M}$. M is the

quotient of a free module F (e.g take the free module with generators the elements of M). Then there is a short exact sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow M \longrightarrow 0.$$

R is a free module, as it is a submodule of a free module. It follows that there is an exact sequence of sheaves

$$0 \longrightarrow \tilde{R} \longrightarrow \tilde{F} \longrightarrow \mathcal{F} \longrightarrow 0.$$

By hypothesis, $\tilde{R}$ and $\tilde{F}$ are free sheaves.
On the other hand, the quotient of a quasi-coherent sheaf is quasi-coherent and a free sheaf is quasi-coherent, so that the quotient of a free sheaf is quasi-coherent.
If $\mathcal{F}$ is coherent then we may assume that F is finitely generated, in which case R is finitely generated. But then $\mathcal{R}$ and $\mathcal{G}$ have finite rank. On the other hand a free sheaf of finite rank is coherent and if X is Noetherian then a quotient of a coherent sheaf is coherent, so that the quotient of a free sheaf of finite rank is coherent.
5.5. (a) Let $X = \mathbb{A}^1 - \{0\}$ and $Y = \mathbb{A}^1$ and let $f\colon X \longrightarrow Y$ be the natural inclusion. Let $\mathcal{F} = \mathcal{O}_X$. $\mathcal{F}$ is clearly coherent. As Y is affine and $\mathcal{G} = f_*\mathcal{F}$ is quasi-coherent, $\mathcal{G} = \tilde{M}$, where $M = K[x, x^{-1}]$ is a $K[x]$-module. As M is not a finitely generated $K[x]$-module, it follows that $\mathcal{G}$ is not coherent.
(b) Since the property of being a closed immersion and being a finite morphism is local on Y, we may assume that $Y = \operatorname{Spec} B$ is affine. In this case, $X = \operatorname{Spec} A$ is affine as it is isomorphic to a closed subset of an affine variety. As f is a closed immersion, A is a quotient of B. It follows that A is a finitely generated B-module, so that f is a finite morphism.
(c) Since the property of being coherent and being a finite morphism is local on Y, we may assume that $Y = \operatorname{Spec} B$ is affine. In this case $X = \operatorname{Spec} A$ is affine and A is a finitely generated B-module. As X is affine, $\mathcal{F} = \tilde{M}$, for some A-module M. Then $f_*\mathcal{F} = \tilde{M}$, but where M is now considered as a B-module. Suppose that B is generated by $b_1, b_2, \dots, b_s$, as an A-module, and that M is generated by $m_1, m_2, \dots, m_t$, as a B-module. Then M is generated by $b_i m_j$, $1 \leq i \leq s$ and $1 \leq j \leq t$. Thus $f_*\mathcal{F}$ is coherent.
5.6 (a) Note that $m \in M$ is zero in $M_{\mathfrak{p}} = \tilde{M}_{\mathfrak{p}}$ if and only if there is an element $a \in A$, not in $\mathfrak{p}$, such that $a \cdot m = 0$. But then $\mathfrak{p} \in \operatorname{Supp} m$ if and only if there $a \in A$, not in $\mathfrak{p}$, such that $a \cdot m \neq 0$, that is, $a \notin \operatorname{Ann} m$.

(b) Pick generators $m_1, m_2, \ldots, m_k$ for the A-module M. Then

$$\operatorname{Ann} M = \sum \operatorname{Ann} m_i.$$

On the other hand, the images of $m_1, m_2, \ldots, m_k$ generate the stalks. It follows that

$$\operatorname{Supp} \mathcal{F} = \bigcap_{i=1}^{k} V(\operatorname{Ann} m_i) = V(\operatorname{Ann} M).$$

(c) One can check this locally, so this follows immediately from (b).
(d) By (II.1.20) $\mathcal{H}^0_Z(\mathcal{F})$ is a subsheaf of $\mathcal{F}$, so that $\mathcal{H}^0_Z(\mathcal{F})$ is quasi-coherent.

$$\Gamma_Z(\mathcal{F}) \subset M,$$

is the subset of sections vanishing on $U = X - Z$. This is clearly $\Gamma_{\mathfrak{a}}(M)$.
(e) One can check this locally, so this follows immediately from (d).