MODEL ANSWERS TO HWK #6

1. (4.1). Suppose that $f\colon X \longrightarrow Y$ is a finite morphism of schemes. Since properness is local on the base, we may assume that $Y = \operatorname{Spec} B$ is affine. By (3.4) it follows that $X = \operatorname{Spec} A$ is affine and A is a finitely generated B-module. It follows that A is integral over B. There are two ways to proceed.

Here is the first. f is separated as X and Y are affine. As A is a finitely generated B-module it is certainly a finitely generated B-algebra and so f is of finite type. Since the property of being finite is stable under base extension, to show that f is universally closed it suffices to prove that f is closed.

Let $I \trianglelefteq A$ be an ideal and let $J \trianglelefteq B$ be the inverse image of I. I claim that $f(V(I)) = V(J)$. One direction is clear, the LHS is contained in the RHS. Otherwise suppose $\mathfrak{q} \in V(J)$, that is, $J \subset \mathfrak{q}$. We want to produce $I \subset \mathfrak{p}$ whose image is $\mathfrak{q}$. Equivalently we want to lift prime ideals of B/J to prime ideals of A/I. But A/I is integral over B/J and what we want is the content of the Going up Theorem in commutative algebra.

Here is the second. Pick $a_1, a_2, \dots, a_n \in A$ which generate A as a B-module. Let $C = B[a_1]$ and let $Z = \operatorname{Spec} C$. Then there are finite morphisms $X \longrightarrow Z$ and $Z \longrightarrow Y$. Since the composition of proper morphisms is proper, we are reduced to the case $n = 1$, by induction. Since A is integral over B, we may find a monic polynomial

$$m(x) = x^d + b_{d-1}x^{d-1} + \dots + b_0 \in B[x],$$

such that $m(a) = 0$. Thus we have a closed immersion $X \subset \mathbb{A}^1_Y$. Let

$$M(X, Y) = X^d + b_{d-1}X^{d-1}Y + \dots + b_0 Y^d \in B[X, Y],$$

be the homogenisation of $m(x)$. Note that the corresponding closed subset of $\mathbb{P}^1_B$ is the same as the closed subset $\mathbb{A}^1_B$, since the coefficient in front of X^d does not vanish. Thus there is a closed immersion $X \longrightarrow \mathbb{P}^1_Y$ and so $X \longrightarrow Y$ is projective, whence proper.

(4.2). Let $h\colon X \longrightarrow Y \underset{S}{\times} Y$ be the morphism obtained by applying the universal property of the fibre product to both f and g. Then the image of h (set-theoretically) must land in the image of the diagonal morphism, as this is true on a dense open subset, and the image of the diagonal is closed. As X is reduced then in fact h factors through the diagonal morphism and so $f = g$.

(a) Let X be the subscheme of $\mathbb{A}^2_k$ defined by the ideal $\langle x^2, xy\rangle$, so that X is the union of the x-axis and the length two scheme $\langle y, x^2\rangle$ (in fact X contains any length 2 scheme with support at the origin). Then there are many morphisms of X into $Y = \mathbb{A}^3_k$ which are the identity on the x-axis. Indeed pick any plane π containing the x-axis. Any isomorphism of $\mathbb{A}^2_k$ which is the identity on the x-axis to the plane π determines a morphism from X, by restriction. Moreover π is the smallest linear space through which this morphism factors. Thus any two such maps are different if we choose a different plane but all such morphisms are the same if we throw away the origin from X.
(b) Let Y be the non-separated scheme obtained by identifying all of the points of two copies of $\mathbb{A}^1_k$, apart from the origins. If p_1 and p_2 are the images of the origins in Y then $Y - \{p_1, p_2\}$ is a copy of $\mathbb{A}^1_k - \{0\}$. This gives us an isomorphism $\mathbb{A}^1_k - \{0\} \longrightarrow Y - \{p_1, p_2\}$ and by composition a morphism $\mathbb{A}^1_k - \{0\} \longrightarrow Y$. Clearly there are two ways to extend the morphism $\mathbb{A}^1_k - \{0\} \longrightarrow Y$ to the whole of $X = \mathbb{A}^1_k$.
(4.3). Consider the commutative diagram

$$\begin{array}{ccc} U \cap V & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ U \underset{S}{\times} V & \longrightarrow & X \underset{S}{\times} X, \end{array}$$

where the bottom arrow is the natural morphism induced by the natural inclusions $i\colon U \longrightarrow X$ and $j\colon V \longrightarrow X$. Suppose that W maps to both X and $U \underset{S}{\times} V$ over $X \underset{S}{\times} X$. Then there are two morphisms to U and V, which become equal when we compose with i and j. Hence the image of this morphism must lie in $U \cap V$ and so this commutative diagram is in fact a fibre square.
As $X \longrightarrow S$ is separated, the diagonal morphism $\Delta\colon X \longrightarrow X \underset{S}{\times} X$ is a closed immersion. As closed immersions are stable under base extension, $U \cap V \longrightarrow U \underset{S}{\times} V$ is a closed immersion. But $U \underset{S}{\times} V$ is affine, since U, V and S are all affine. (II.3.11) implies that every closed subset of an affine scheme is affine and so $U \cap V$ is affine.
Let $S = \operatorname{Spec} k$, where k is a field and let Y be the non-separated scheme obtained by taking two copies of $\mathbb{A}^2_k$ and identifying all of their points, except the origins. Then Y contains two copies U and V of $\mathbb{A}^2_k$, both of which are open and affine. However, the intersection $U \cap V$ is a copy of $\mathbb{A}^2_k - \{0\}$, which is not affine.
(4.5). (a) We apply the valuative criteria for separatedeness. Let $T = \operatorname{Spec} R$ and $U = \operatorname{Spec} K$. Then there is a morphism $U \longrightarrow X$, obtained

by sending t_1 to the generic point of X. Suppose that the valuation has centres x and $y \in X$. Then R dominates both $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ and by (II.4.4) there are two morphisms $T \longrightarrow X$ obtained by sending t_0 to x or y. As X is separated these two morphisms are the same by the valuative criteria. In particular $x = y$ and the centre of every valuation of K/k is unique.
(b) Since proper implies separated, uniqueness follows from (a). Once again there is a morphism $U \longrightarrow X$. By the valuative criteria for properness this gives a morphism $T \longrightarrow X$. By (II.4.4) if x is the image of x_0 then R dominates $\mathcal{O}_{X,x}$. But then x is the centre of the corresponding valuation.
(c) First some generalities about valuations. Let R be a valuation ring in the field L. Suppose we are given a diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow f \\ T & \longrightarrow & \operatorname{Spec} k, \end{array}$$

where $U = \operatorname{Spec} L$ and $T = \operatorname{Spec} R$. By (II.4.4) $k \subset R$ so that R is a valuation ring of L/k. Let x_1 be the image of t_1 and let Z be the closure of x_1, with the reduced induced structure. Let M be the function field of Z. By (II.4.4) we are given an inclusion $M \subset L$. Let $R' = R \cap M \subset M$. It is easy to see that R' is a local ring. As M is a quotient of $\mathcal{O}_{X,x_1}$, we can lift R' to a ring $S' \subset \mathcal{O}_{X,x_1} \subset K$. Finally, by Zorn's Lemma, we may find a local ring $S \subset K$ containing S', maximal with this property, so that S is a valuation ring of K/k.
Now suppose that every valuation of K/k has at most one centre on X. Suppose we are given two morphisms $T \longrightarrow X$. Let x and y be the images of t_0. By (II.4.4) we are given inclusions $\mathcal{O}_{Z,x} \subset R$ and $\mathcal{O}_{Z,y} \subset R$. So $\mathcal{O}_{Z,x} \subset R'$ and $\mathcal{O}_{Z,y} \subset R'$. $\mathcal{O}_{Z,x}$ and $\mathcal{O}_{Z,y}$ lift to $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ in $\mathcal{O}_{X,x_1}$, so that $\mathcal{O}_{X,x} \subset S'$ and $\mathcal{O}_{X,y} \subset S'$. Thus $\mathcal{O}_{X,x} \subset S$ and $\mathcal{O}_{X,y} \subset S$ so that x and y are two centres of S. But then $x = y$ by hypothesis and so the valuative criteria implies X is separated.
Now suppose that every valuation of K/k has a unique centre on X. By hypothesis S has a centre x on X. In this case $\mathcal{O}_{X,x} \subset S$. $x \in Z$ so that in fact $\mathcal{O}_{X,x} \subset S'$. It follows that $\mathcal{O}_{Z,x} \subset R' \subset R$. By (II.4.4) this gives us a lift $T \longrightarrow X$ and by the valuative criteria X is proper over k.
(d) Suppose not. Then we may find $a \in \Gamma(X, \mathcal{O}_X)$ such that $a \notin k$. Then $1/a \in K$ is not in k. As k is algebraically closed, $k[1/a]$ is isomorphic to a polynomial ring and $k[1/a]_{1/a}$ is a local ring. By Zorn's Lemma there is a ring R such that $1/a \in \mathfrak{m}_R$ and R is maximal with

respect to domination, that is, R is a valuation ring. As X is proper, R has a unique centre x on X. Thus $a \in \mathcal{O}_{X,x} \subset R$ so that $a \in R$. This contradicts the fact that $1/a \in \mathfrak{m}_R$.
(4.6). Suppose that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. First assume that X and Y are reduced, that is A and B have no nilpotents. Note that X is Noetherian as X is of finite type over k. Let K be the field of fractions of A. Let $R \subset K$ be a valuation ring which contains B.
By (II.4.4) we get a diagram

$$\begin{array}{ccc} U & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ T & \longrightarrow & Y. \end{array}$$

As f is proper, it follows that we may find a morphism $T \longrightarrow X$ making the diagram commute. Let $x \in X$ be the image of $x_0 \in T$. By (II.4.4) $A \subset \mathcal{O}_{X,x} \subset R$. Since this is true for every R, (II.4.11A) implies that A is contained in the integral closure of B inside L. But then A is a finitely generated B-module, as it is a finitely generated B-algebra.
We now prove the general case. Note that the following commutative diagram

$$\begin{array}{ccc} X\text{red} & \longrightarrow & X \\ {\scriptstyle f\text{red}}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y\text{red} & \longrightarrow & Y, \end{array}$$

is a fibre square. So $f$red is a proper morphism. Now $X\text{red} = \operatorname{Spec} A/I$ and $Y\text{red} = \operatorname{Spec} B/J$, where I and J are the ideals of nilpotent elements and by what we have already proved A/I is a finite B/J-module. It follows that A/I is an integral extension of B/J. This implies that A/I is integral over B. As A is integral over A/I (the polynomial $x^n \in A[x]$ is monic) it follows that A is integral over B. As A is finitely generated B-algebra it follows that A is a finitely generated B-module.
2. **Challenge Problems** (4.10). (a) Assume that this result holds if X is irreducible.
As X is of finite type over S and S is Noetherian, X is Noetherian. It follows that X is a finite union of irreducible components $X_1, X_2, \dots, X_k$. As the natural inclusion $X_i \longrightarrow X$ is a closed immersion, a closed immersion is proper and the composition of proper morphisms is proper, it follows that the natural morphism $X_i \longrightarrow S$ is proper. By hypothesis, for each i, we may find a morphism $g_i \colon X'_i \longrightarrow X_i$ and an open subset $U_i \subset X_i$ such that $g_i^{-1}(U_i) \longrightarrow U_i$ is an isomorphism and $X'_i \longrightarrow S$ is projective. Let X' be the disjoint union of the X'_i. Then the natural morphism $X' \longrightarrow S$ is projective.

On the other hand, let X_0 be the disjoint union of $X_1, X_2, \dots, X_k$. Then there are natural morphisms, $X' \longrightarrow X_0$ and $h\colon X_0 \longrightarrow X$. There is an open subset V of X such that $h^{-1}(V) \longrightarrow V$ is an isomorphism. Let U_0 be the union of U_i and let U be the image of $U_0 \cap h^{-1}(V)$. Then U is an open subset of X (indeed it is an open subset of V) and if $g\colon X' \longrightarrow X$ denotes the composition, then $g^{-1}(U) \longrightarrow U$ is an isomorphism.

(b) As S is Noetherian we can cover S by finitely many open affines $S_1, S_2, \dots, S_k$. As f is of finite type we can cover $f^{-1}(S_j = \operatorname{Spec} A)$ by finitely many open affines $U_i = \operatorname{Spec} B \subset X$, $U_1, U_2, \dots, U_n$ where B is a finitely generated A-algebra. If we pick generators $a_1, a_2, \dots, a_m$ for A as a A-algebra then we get a surjective rimg homomorphism

$$A[x_1, x_2, \dots, x_m] \longrightarrow B.$$

Thus there is a closed immersion $U_i \longrightarrow \mathbb{A}^m_{S_j}$. As there is an open immersion $S_i \longrightarrow S$ it follows that there is an open immersion $\mathbb{A}^m_{S_i} \longrightarrow \mathbb{A}^m_S$. Composing with the open immersion $\mathbb{A}^m_S \longrightarrow \mathbb{P}^m_S$, we get an open immersion $\mathbb{A}^m_{S_i} \longrightarrow \mathbb{P}^m_S$. Taking the closure P_i of the image of U_i in $\mathbb{P}^m_S$, we get an open immersion $U_i \longrightarrow P_i$, where P_i is projective over S. In particular $U_1, U_2, \dots, U_n$ are quasi-projective over S.

(c) Note that $h\colon X' \longrightarrow P$ is proper as $P \longrightarrow S$ is separated and the composition $X' \longrightarrow S$ is proper (see (d) of (II.4.8)). In particular h is closed. Given $x' \in X'$ let $x \in X$ be the image. Then $x \in U_i$, some $1 \leq i \leq n$. Let $p \in P$ be the image of x in P and let p_i be the image of p in P_i under the natural projection. By assumption the induced morphism $h_i\colon X' \longrightarrow \mathcal{O}_{P_i}$ is a closed immersion in a neighbourhood of x. Since closed immersions satisfy properties (a-c) of (II.4.8) it follows that h is closed immersion in a neighbourhood of x. Therefore h is a closed immersion.

(d) As the natural morphism $U \longrightarrow U_i$ is an open immersion and the composition of open immersions is an open immersion, it follows that $U \longrightarrow X$ and $U \longrightarrow P_i$ are all open immersions. But then $U \longrightarrow P$ is an isomorphism onto its image and $g^{-1}(U) \longrightarrow U$ is an isomorphism.

(4.11)