MODEL ANSWERS TO HWK #5

1. Suppose that the two dimensional vector space corresponding to l_i is spanned by u_i and v_i. Let l be a line that meets l_1 at p and l_2 at q. As $p \in l_1$ and $q \in l_2$, l is represented by $\omega = (a_1u_1 + b_1v_1) \wedge (a_2u_2 + b_2v_2)$. Expanding, ω is a combination of $u_1 \wedge u_2$, $u_1 \wedge v_2$, $v_1 \wedge u_2$ and $v_1 \wedge v_2$. Let U be the span of these four vectors. In particular the locus of lines which meets l_1 and l_2 is certainly a subset of $\mathbb{P}(U)$. But the condition that any such form is decomposable, is equivalent to the condition that it is of the form $\omega = (a_1u_1 + b_1v_1) \wedge (a_2u_2 + b_2v_2)$. If we expand ω then we get the standard embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ (up to change of sign).

Alternatively it is clear that abstractly the locus of lines meeting l_1 and l_2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, as a line is specified by its intersection with l_1 and l_2.

If l_1 and l_2 intersect, then a line that meets both of them is ether a line that contains $p = l_1 \cap l_2$ or a line contained in the plane $H = \langle l_1, l_2\rangle$. Thus the locus of lines is the union $\Sigma_p \cup \Sigma_H$, which we have seen is the union of two planes. There locus of lines which meet p and are contained in H is a line. So $\Sigma_p \cup \Sigma_H$ is the union of two planes meeting along a line.

2. The point is that there is no moduli to this question, so that we are free to choose our favourite quadric. If we take $XW = YZ$, so that we have the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the morphism

$$([X_0 : X_1], [Y_0 : Y_1]) \longrightarrow [X_0Y_0 : X_1Y_0 : X_0Y_1 : X_1Y_1],$$

then the two families of lines are

$$[aS : aT : bS : bT] \qquad \text{and} \qquad [aS : bS : aT : bT],$$

where the pair $[a : b]$ parametrises the two families, and $[S : T]$ parametrises the lines themselves (for fixed $[a : b]$). Thus a general line from the first family is the span of

$$[a : 0 : b : 0] \qquad \text{and} \qquad [0 : a : 0 : b],$$

whilst a general line from the second family is the span of

$$[a : b : 0 : 0] \qquad \text{and} \qquad [0 : 0 : a : b].$$

Thus a line from the first (respectively second family) is represented by

$$\omega = (ae_1 + be_3) \wedge (ae_2 + be_4) \qquad \text{respectively} \qquad (ae_1 + be_2) \wedge (ae_3 \wedge be_4).$$

Expanding, the family of lines from the first family is given as

$$a^2(e_1 \wedge e_2) + ab(e_1 \wedge e_4 + e_3 \wedge e_2) + b^2(e_3 \wedge e_4),$$

and the second is given as

$$a^2(e_1 \wedge e_3) + ab(e_1 \wedge e_4 + e_2 \wedge e_3) + b^2(e_2 \wedge e_4).$$

Thus we get two conics lying in the two planes spanned by $e_1 \wedge e_2$, $e_1 \wedge e_4 + e_3 \wedge e_2$ and $e_3 \wedge e_4$, and $e_1 \wedge e_3$, $e_1 \wedge e_4 + e_2 \wedge e_3$ and $e_2 \wedge e_4$. Since these vectors span $\wedge^2 V$, the two planes are complementary, and neither of them is contained in $\mathbb{G}(1,3)$.

Now suppose that we have a plane conic $C \subset \mathbb{G}(1,3)$, where the span Λ of C, is not contained in $\mathbb{G}(1,3)$. In this case, by reasons of degree, $C = \Lambda \cap \mathbb{G}(1,3)$.

Suppose that when we take two general points of the conic the corresponding lines l and m intersect in $\mathbb{P}^3$. Pick a third point, corresponding to a third line n. If there is a common point p to all three then the conic C meets the plane Σ_p in three points, so that the conic C must contain the line $\Sigma_p \cap \Lambda$, a contradiction. But then l, m and n must be coplanar (they lie in the plane spanned H by the three intersection points $m \cap n$, $l \cap n$ and $l \cap m$). In this case C contains three points of the plane Σ_H, so that it contains the line $\Lambda \cap \Sigma_H$, a contradiction.

So now we know that two general points of C correspond to two skew lines. There are two ways to finish. Here is the first. We may find three points of C which correspond to three skew lines l, n and m. Three skew lines have no moduli, that is, any three skew lines are projectively equivalent (proved in class), so there is an element $\phi \in \mathrm{PGL}_4(K)$ which carries these three lines to any other three. ϕ acts on $\mathbb{P}(\wedge^2 V)$, fixing $\mathbb{G}(1,3)$ and carries three points of the plane Λ to any other three points of $\mathbb{G}(1,3)$ which correspond to three skew lines. But any plane is determined by any three points which are not collinear and so we may assume that Λ is the plane coming from the quadric, as above.

Here is the second. $\mathbb{G}(1,3)$ is determined by a quadratic polynomial of maximal rank. This determines a bilinear form on $\wedge^2 V$ (up to scalars). In particular given Λ there is a dual plane Λ', which is complementary to Λ and is also not contained in $\mathbb{G}(1,3)$. Let $C' = \Lambda' \cap \mathbb{G}(1,3)$, another smooth conic. Since Λ' is dual to Λ under the pairing determined by $\mathbb{G}(1,3)$ this says that if we pick $[u \wedge v] \in C$ and $[u' \wedge v'] \in C'$ then $u \wedge v \wedge u' \wedge v' = 0$, that is, the corresponding lines l and l' are concurrent.

So now we have two families of skew lines $\{l\}$ and $\{l'\}$ in $\mathbb{P}^3$, such that a pair of lines from both families are concurrent. Pairs of lines from both families are parametrised by $\mathbb{P}^1 \times \mathbb{P}^1$ and we get a morphism

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3,$$

which sends pair (l, m) to $l \cap m$. This morphism has a bidegree, which must be $(1, 1)$ since $\mathbb{P}^1 \times \{p\}$ and $\{q\} \times \mathbb{P}^1$ are both sent to a line. But then the image is the Segre, up to projective equivalence and C is just the family of lines of one ruling.

If Λ is contained in $\mathbb{G}(1, 3)$ then $\Lambda = \Sigma_p$ or Σ_H. In the first case, a conic in Σ_p is the same as the family of lines in a quadric cone (which automatically pass through the vertex p of the cone). If $\Lambda = \Sigma_H$, then a conic $C \subset \Lambda$ is simply the family of tangent lines to a conic in H.

3. Let's warm up a little and see what happens if we start with the line m given by $Z_2 = Z_3 = 0$. Note that for each point p of this line we get a plane $\Sigma_p \subset \mathbb{G}(1, 3)$. So we want a family of planes inside $\mathbb{G}(1, 3)$. The natural guess is that this family is given by a hyperplane section. If we look at the hyperplane section $p_{34} = 0$ we get a cone over a quadric in $\mathbb{P}^3$. This is indeed covered by copies of $\mathbb{P}^2$. The condition that $p_{34} = 0$ means that that the term $e_3 \wedge e_4$ does not appear, which is the condition that we meet the line m.

(a) Since a conic degenerates to a union of two intersecting lines, the equation defining this conic ought to be quadratic. Consider $\lambda Z_1^2 - \mu Z_0 Z_2$. If we let λ go to zero then we get $Z_0 Z_2 = 0$, the union of two lines. This gives the equation $p_{14} p_{34}$. On the other hand if we let μ go to zero we get the line $Z_1^2 = 0$ counted twice. This gives the equation $p_{24}^2 = 0$. So we guess the equation we want is some linear combination of $p_{14} p_{34}$ and $p_{24}^2 = 0$. Let's guess

$$p_{14} p_{34} = p_{24}^2.$$

Now an open subset of points of the conic has the form $[t^2 : t : 1 : 0]$. Thus an open subset of the points of the Grassmannian which intersect this conic has the form

$$\begin{pmatrix} t^2 & t & 1 & 0 \\ 0 & a & b & 1 \end{pmatrix}.$$

We have $p_{14} = t^2$, $p_{34} = 1$ and $p_{24} = t$. Clearly these set of points satisfy the equation $p_{14} p_{34} = p_{24}^2$. Now suppose we start with a line l whose Plücker coordinates satisfy this equation. Let $A = (a_{ij})$ be a 2×4 matrix whose rows span the plane corresponding to l. If the last column is zero then $p_{i4} = 0$ and the equation holds automatically. Applying elementary row operations, we may assume that the last column is the vector $(0, 1)$. In this case $p_{i4} = a_{1i}$ and the first row has the form

$(t^2, t, 1, 0)$ or it is equal to $(1, 0, 0, 0)$. Either way, this corresponds to a point on the conic.
(b) Recall that the ideal of the twisted cubic C is generated by the three quadrics $Q_0 = Z_0Z_3 - Z_1Z_2$, $Q_1 = Z_1^2 - Z_0Z_2$, $Q_2 = Z_2^2 - Z_1Z_3$. Now note that a line l intersects the twisted cubic if and only if the restrictions of Q_0, Q_1 and Q_2 to l span a vector space of dimension at most two.
Indeed if the line l intersects C then $q_i = Q_i|_l$ all have a common zero and so cannot span the full space of quadratic polynomials on l, which has dimension three (and no common zeroes). Conversely if q_0, q_1 and q_2 span a vector space of dimension at most two then some linear combination $Q = \lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2$ contains the line l. In this case l is a line of one of the rulings of Q, C is a curve of type $(2, 1)$ on $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and so l intersects C in one (or two) point(s).
Consider the open subset U of the Grassmannian where $p_{12} = 1$, that is consider matrices of the form

$$A = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

Natural coordinates on any line $l \in U$ are $X = Z_0$ and $Y = Z_1$. In fact at the point $[\lambda : \mu : \lambda a + \mu c : \lambda b + \mu d]$ of the line, we have

$$\begin{aligned} Z_0 &= \lambda = X \\ Z_1 &= \mu = Y \\ Z_2 &= \lambda a + \mu c = aX + cY \\ Z_3 &= \lambda b + \mu d = bX + dY. \end{aligned}$$

In this basis

$$\begin{aligned} q_0 &= bX^2 + (d - a)XY - cY^2 \\ q_1 &= -aX^2 - cXY + Y^2 \\ q_2 &= a^2X^2 - (2ac - b)XY + (c^2 - d)Y^2. \end{aligned}$$

It follows that the locus where are interested in is the rank two locus of the following matrix

$$\begin{pmatrix} b & d - a & -c \\ -a & -c & 1 \\ a^2 & 2ac - b & c^2 - d \end{pmatrix}.$$

If we expand this determinant then we get

$$-ad^2 + ac^2d + bcd + 2a^2d - bc^3 - 3abc + b^2 - a^3.$$

Note that $e = ad - bc$ is a determinant. Thus the term of degree four simplifies to

$$ac^2d - bc^3 = c^2(ad - bc) = c^2e.$$

Note that $a = -p_{23}/p_{12}$, $b = -p_{24}/p_{12}$, $c = p_{13}/p_{12}$, $d = p_{14}/p_{12}$, and $e = p_{34}/p_{12}$. Substituting and multiplying by p_{12}^3 gives an equation of degree three in the Plücker coordinates,

$$p_{13}^2p_{34} + p_{23}p_{14}^2 - p_{24}p_{13}p_{14} + 2p_{23}^2p_{14} - 3p_{23}p_{24}p_{13} + p_{12}p_{24}^2 + p_{23}^3.$$