MODEL ANSWERS TO HWK #4

1. We may as well assume that the first three points are 0, ∞, 1, and in this case the fourth point is the cross-ratio λ. Moreover it suffices, by symmetry, to produce an element of PGL(2) that switches 0 and ∞ and 1 and λ. Now elements of PGL(2) that switch 0 and ∞ are of the form $z \longrightarrow \frac{a}{z}$. Under this map, 1 is sent to a. So we set $a = \lambda$. Clearly λ is sent to one, as required.

2. We have already seen that any function of the four points p_1, p_2, p_3 and p_4 which is invariant under the action of PGL(2), is in fact a rational function of λ. In other words, we want to determine the fixed field of $L = K(\lambda)$ under the induced action of S_4.

Now the quotient S_4/V is isomorphic to S_3. Thus the orbit of any set of four points, up to isomorphism and relabelling, is in fact an orbit of S_3. So in fact we only need the fixed field under S_3. Now any two transpositions generate S_3. The transposition $(1,2)$ is induced by $z \longrightarrow 1/z$ (this switches 0 and ∞ and fixes 1). Under this map, λ is sent to $1/\lambda$. Similarly the map $z \longrightarrow 1-z$ induces the transposition $(1,3)$, since it switches 0 and 1, but fixes ∞. This map sends λ to $1-\lambda$. By standard Galois theory, if M is the fixed field, so that $L/M/K$, then the extension L/M has degree six. Let $N = K(j)$. Note first that j is invariant under the two maps

$$\lambda \longrightarrow 1/\lambda, \qquad \text{and} \qquad \lambda \longrightarrow 1-\lambda.$$

This says that $L/N/M$, that is, N is intermediary between L and M. On the other hand, it is a standard result in a first course on Galois theory, that if $L = K(x)$, where x is transcendental, and $N = K(f(x)/g(x))$ then the degree of the extension L/N is precisely the maximum degree of f and g. In our case both f and g have degree six, so that L/N has degree six. It follows that $N = M$, as required.

The j-invariant extends to a morphism $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$. This morphism is surjective and also sends 0, 1 and ∞ to ∞. It follows that the j-invariant takes values in $\mathbb{A}^1$.

3. Typically $G = V$ so that the quotient is trivial. There are two other possibilities. It is possible that there is an extra involution. For example, $z \longrightarrow 1/z$ fixes both 1 and -1, so that the four points 0, 1, ∞ and -1 have an extra involution. In this case the j-invariant is

$$2^8 \frac{(1+1+1)^3}{1(-1-1)^2} = 1728.$$

The other possibility is that there is an extra 3-cycle. For example, 1, ω, ω^2 and ∞ are fixed under $z \longrightarrow \omega z$. Subtracting 1, we get 0, $\omega - 1$, $\omega^2 - 1$ and ∞. Dividing through by $\omega - 1$ we get 0, 1, $1 + \omega$ and ∞. The j-invariant is

$$2^8 \frac{((\omega+1)^2 - (1+\omega) + 1)^3}{(1+\omega)(\omega))^2} = 2^8 \frac{(\omega^2+\omega+1)^3}{(1+\omega)(\omega))^2} = 0.$$

Now there are no configuration of points with anymore symmetries, since then $G = S_4$ and the j-invariant would have to be both 1728 and 0, impossible.

4. Let $p_1, p_2, \dots, p_n$ be n points in $\mathbb{P}^1$. We denote an unordered set of points by the formal sum

$$D = p_1 + p_2 + \cdots + p_n.$$

Associate to D the polynoial of degree n

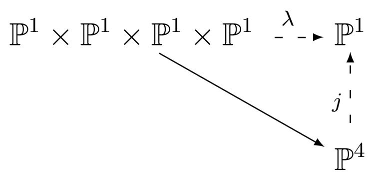
$$F(X, Y),$$

whose zeroes are given by D (clearly the zeroes of a polynomial are unordered). Note that D is determined by the equivalence class of F up to scalars, that is, a point of

$$\mathbb{P}(\mathrm{Sym}^n(V^*))$$

Since the latter space is isomorphic to $\mathbb{P}^n$, the result follows.

5. Note that we have a commutative diagram



Here the diagonal map is simply the natural map which associates to an ordered pair $(p_1, p_2, \dots, p_n)$ an unordered pair $D = p_1 + p_2 + \cdots + p_n$. In other words, to show that the rational map j extends across the big diagonal (where two points come together) it suffices to prove that the cross-ratio extends across a component of the big diagonal, and that the value we get is independent of re-ordering.

Thus we may assume that p_1, p_2, p_3 are distinct and p_4 is approaching p_3. Note that as a map to $\mathbb{P}^1$, the cross-ratio is given as

$$(p_1, p_2, p_3, p_4) \longrightarrow [(p_1 - p_4)(p_2 - p_3) : (p_1 - p_3)(p_2 - p_4)]$$

It follows that the morphism is well defined, since neither factor is zero when $p_3 = p_4$. In fact it is clear that under these circumstances the cross-ratio approaches $[1 : 1]$. The only other possibilities are that the

cross-ratio approaches either $[0:1]$ or $[1:0]$. If $\lambda = 0$, 1, or ∞, then $j = \infty$.
So as two points approach each other, the j-invariant approaches ∞.
6. Suppose that the point $p = [v]$ and that the plane H corresponds to $W \subset V$. Then a line l containing p, contained in H is spanned by the vector v and a vector $w \in W$, so that as a point of $\mathbb{P}(\bigwedge^2 V)$, $[l] = [\omega] = [v \wedge w]$. Now if W has basis v, w_1, w_2, then we can choose $w = aw_1 + bw_2$, so that vector ω lies in the plane $v \wedge w_1$ and $v \wedge w_2$; indeed $\omega = av \wedge w_1 + bv \wedge w_2$. But this corresponds to a line L in $\mathbb{P}^5$, lying on the Grassmannian.
Now suppose that we have a line L in $\mathbb{P}^5$, lying on the Grassmannian. Any such line consists of a family $\omega = a\omega_1 + b\omega_2$ of decomposable forms, so that $\omega_i = u_i \wedge v_i$. Now if the span of the vectors u_1, u_2, v_1 and v_2 is the whole of V, then $\omega_1 + \omega_2$ is indecomposable. Otherwise v_2 is a linear combination of u_1, u_2 and v_1, so that L parametrises lines in W, the span of u_1, u_2, and v_1. But then ω_1 and ω_2 must be divisible by the same vector v (for example, by duality). Thus $p = [v]$ and $H = \mathbb{P}(W)$.
7. Suppose $p = [v]$. If the line l contains p, then it may be represented by $\omega = v \wedge w$. Suppose that we extend v to a basis v, w_1, w_2, w_3. Then we may assume that $w = a_1w_1 + a_2w_2 + a_3w_3$, so that l is represented by $a_1\omega_1 + a_2\omega_2 + a_3\omega_3$, where $\omega_i = v \wedge w_i$. Σ_p is the corresponding plane.
Now suppose that $H = \mathbb{P}(W)$. Pick a basis w_1, w_2, w_3 for W. Then a line l in H is represented by a form $\omega = a_1 w_2 \wedge w_3 + a_2 w_3 \wedge w_1 + a_3 w_1 \wedge w_2$. Since any rank two from in a three dimensional space is automatically decomposable, the result follows easily. Alternatively, lines contained in H are the same as lines containing $[H]$ in the dual projective space.
Another way to proceed, in either case, is as follows. Consider the surface $P = \Sigma_H$. Pick any two points $[l]$ and $[m] \in P$. Then l and m are two lines in $\mathbb{P}^3$, which are contained in H. Then l and m must intersect and we set $p = l \cap m$. Then we get a line $L = \Sigma_p \cap \Sigma_H = \Sigma_{p,H} \subset P$, by 6, which contains the original two points $[l]$ and $[m] \in L$. It follows that through every two points of the surface P, we may find a unique line L. It follows easily that P is a plane. Similarly for Σ_p.
Now suppose that we are given a plane P inside $\mathbb{G}(1,3) \subset \mathbb{P}^5$. By 6, if $L \subset P$ is a line then there is a point $p \in \mathbb{P}^3$ and a plane $H \subset \mathbb{P}^3$ such that $L = \Sigma_{p,H}$. Suppose that we can find three lines $L_i = \Sigma_{p_i,H_i} \subset P$, $i = 1$, 2 and 3, which form a triangle $\triangle$, such that $\{p_1, p_2, p_3\}$ has cardinality three. Let $l_{ij} \subset \mathbb{P}^3$ be the line corresponding to the intersection point $L_i \cap L_j$. Then $l_{ij} = \langle p_i, p_j \rangle$. In particular p_1, p_2 and p_3 are not collinear so that they span a plane $H = \langle p_1, p_2, p_3 \rangle$. If

$H \neq H_i$ then $l_{ij} = H \cap H_i$, for $j \neq i$, a contradiction (l_{ij} must depend on j). Thus $H_1 = H_2 = H_3 = H$. Now let $L = \Sigma_{q,K} \subset P$ be an arbitrary line. Suppose that $K \neq H$. If m is the line corresponding to a point where L meets the triangle $\triangle$ then $m = H \cap K$. Since L meets the triangle $\triangle$ in at least two points, this is a contradiction. Thus $K = H$ and $P = \Sigma_H$.

It remains to deal with the case that there is no such triangle. Note that the map

$$f \colon \check{P} \longrightarrow \mathbb{P}^3,$$

which assigns to the line $L \subset P$ the point $p \in \mathbb{P}^3$, where $L = \Sigma_{p,H}$, is a morphism. If this map is not constant then it is easy to find a triangle such that $\{p_1, p_2, p_3\}$ has cardinality three. But if f is constant then $P = \Sigma_p$.