MODEL ANSWERS TO HWK #3

1 (i) The images are closed subsets, as $\mathbb{P}^1$ is a projective variety and ν is a morphism.
(ii)

$$(S^4 - \beta S^3 T)(\alpha S T^3 - T^4) = -(\alpha\beta + 1) S^4 T^4 + a S^5 T^3 + b S^3 T^5$$

and

$$(S^3 T - \beta S^2 T^2)(\alpha S^2 T^2 - S T^3) = -(\alpha\beta + 1) S^4 T^4 + a S^5 T^3 + b S^3 T^5.$$

Thus the image lies in the quadric $V(XW - YZ)$.
(iii) To determine the type of the image C, it suffices to determine how C meets both lines of the ruling. If it has type (a, b) then it will meet the lines of one ruling in a points and it will meet the lines of the other ruling in b points.
Now the lines of one ruling are given as $\mu X = \lambda Y$, $\mu Z = \lambda W$. This gives two set of equations for S and T. In the first set, we can factor out $S^2(S - \beta T)$ and in the second set we can factor out $T^2(\alpha S - T)$, and we get the solution $\mu S = \lambda T$.
The lines of the other ruling are given as $\mu X = \lambda Z$ and $\mu Y = \lambda W$. This gives two sets of equations for S and T. In the first set we can factor out S and in the second T. In both cases the equations reduce to

$$\mu S^2(S - \beta T) = \lambda T^2(\alpha S - T).$$

Thus our curve has type $(1, 3)$.
(iv) Let's first figure out the equation of C in bihomogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. The general bihomogeneous polynomial of type $(3, 1)$ looks like $Q_0 X_0 + Q_1 X_1$ where Q_0 and Q_1 have degree three in Y_0 and Y_1. Matching with the lines we found in part (iii), the equation is

$$Y_1^2(\alpha Y_0 - Y_1) X_0 - Y_0^2(Y_0 - \beta Y_1) X_1.$$

To find equations of cubics containing C, we top up this equation by multiplying through by X_0^2 and X_1^2. Note that

$$[X : Y : Z : W] = [X_0 Y_0 : X_1 Y_0 : X_0 Y_1 : X_1 Y_1].$$

Topping up with X_0^2

$$Y_1^2 X_0^2(\alpha X_0 Y_0 - X_0 Y_1) - X_0^2 Y_0^2(X_1 Y_0 - \beta X_1 Y_1) = Z^2(\alpha X - Z) - X^2(Y - \beta W).$$

and topping up with X_1^2

$$X_1^2 Y_1^2(\alpha X_0 Y_0 - X_0 Y_1) - X_1^2 Y_0^2(X_1 Y_0 - \beta X_1 Y_1) = W^2(\alpha X - Z) - Y^2(Z - \beta W).$$

2. (i) Note that PGL(2) is generated by translations $z \longrightarrow z + a$ and $z \longrightarrow 1/z$. Translation leaves each difference unchanged and so the cross-ratio is clearly unchanged. On the other hand

$$\frac{(z_1^{-1} - z_4^{-1})(z_2^{-1} - z_3^{-1})}{(z_1^{-1} - z_3^{-1})(z_2^{-1} - z_4^{-1})} = \frac{(z_4 - z_1)(z_3 - z_2)}{(z_3 - z_1)(z_4 - z_2)} = \lambda.$$

so that the cross-ratio is also invariant under $z \longrightarrow 1/z$. Thus the cross-ratio is invariant under PGL(2).
(ii) Recall that there is a unique element ϕ of PGL(2) carrying the first set of three points to 0, ∞ and 1. Thus the only invariant of four ordered points is the position of the fourth point μ. But the cross-ratio is invariant under the action of PGL(2), and for 0, ∞, 1 and μ it comes out to be

$$\lambda = \frac{(0 - \mu)(1/0 - 1)}{(0 - 1)(1/0 - \mu)} = \mu.$$

3. We know that our curves lie in a unique copy of $\mathbb{P}^1 \times \mathbb{P}^1$ and that they have type $(1, 3)$. One family of lines will meet the curve in one point and all but finitely many of the other lines will meet the curve in three points.
I claim that four lines of the other family will meet the curve in two points. In fact this is given by the zeroes of the discriminant of the polynomial of type $(3, 1)$ with respect to the variables which give the cubic, and the discriminant of a cubic is a quartic polynomial (a more sophisticated way to see there are four such lines proceeds as follows; projection onto the correpsponding factor defines a three to one cover of $\mathbb{P}^1$ and by Riemann-Hurwitz this cover has four branch points [assuming that the ramification points are simple]).
Now four points in $\mathbb{P}^1$ have a one dimension moduli, called the j-invariant (the j-invariant is an invariant of four unordered points; it can be calculated by ordering the points, and taking the cross-ratio). It suffices then to check that the cross-ratio is different, for different values of α and β. Clearly there is a morphism

$$j \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

which assigns to a point $[\alpha : \beta]$ the j-invariant of the four branch points. In fact it suffices, then, to show that this morphism is not constant.
Now the four branch points, are given by the four values of λ and μ such that

$$\mu S^2(S - \beta T) = \lambda T^2(\alpha S - T).$$

has a repeated root. Two such values are $\lambda = 0$ (when the double root is $S^2 = 0$) and $\mu = 0$ (when the double root is $T^2 = 0$). Now if $\beta = 0$, then $\lambda = 0$ represents a point where there is a triple root. But there are values of α and β for which we get four distinct roots. Therefore j is not constant, and so it must be surjective.

4. The trick here is instead of computing the equations of the twisted cubic, projecting, and then computing resultants, instead compute the composite morphism and write down the obvious equation satisfied by the image. Projection from $[1 : 0 : 0 : 1]$ is given by the morphism $[X : Y : Z : W] \longrightarrow [X - W : Y : Z]$ (indeed the crucial point is that $[1 : 0 : 0 : 1]$ is the unique point of indeterminancy). Thus the composite morphism is

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

given by

$$[S : T] \longrightarrow [S^3 - T^3 : S^2T : ST^2]$$

Thus on an affine open set, we have

$$t \longrightarrow (1 - t^3/t, t) = (x, z).$$

Thus $xz = 1 - x^3$ is the equation of the image. On $\mathbb{P}^2$ we have $XYZ = Y^3 - X^3$.

Similarly, projection from $[0 : 1 : 0 : 0]$ is given by $[X : Y : Z : W] \longrightarrow [X : Z : W]$, so that the morphism on $\mathbb{P}^1$ is given as

$$[S : T] \longrightarrow [S^3 : ST^2 : T^3]$$

Thus on an affine piece we have

$$t \longrightarrow (t^2, t^3)$$

which we know is the equation of the cuspidal cubic, $Y^2Z = X^3$.

On the other hand, we know that if $\mathbb{P}^1 = \mathbb{P}(V)$, then $\mathbb{P}^3 = \mathbb{P}(\mathrm{Sym}^3(V))$. Now, under the action of PGL(2), $\mathbb{P}^3$ splits into three orbits. In fact, thinking of an element of $\mathrm{Sym}^3(V)$ as being a polynomial of degree three, the orbits are classified by the positions of the roots. If there are three roots, then under the action of PGL(2), we can think of these roots as 0, 1 and ∞. In this case the corresponding polynomial is $XY(X + Y)$. If there are two roots, one of them necessarily double, we can assume the double root is at 0 and the other at ∞. In this case we get the polynomial X^2Y. Finally if there is one root, we can assume it is at ∞, and the corresponding polynomial is X^3. The last case corresponds to points of the twisted cubic itself. Thus there are two orbits not on the twisted cubic.

5. This follows from general principles; the morphism ν is given by quartic polynomials in S and T. On the other hand it is not hard to see that if one composes the morphism

$$[S : T] \longrightarrow [S^4 : S^3T : S^2T^2 : ST^3 : T^4],$$

with projection from the point $[\beta^2 : -\beta : 1 : -\alpha : \alpha^2]$ then one gets the morphism ν.

6. Note first that the space of all $m \times n$ matrices over the field K, is naturally represented by the affine space $X = \mathbb{A}^{mn}$. Setting $m = n$, a moments thought will convince the reader that if we expand

$$\det(A - tI)$$

then we we get a polynomial $f(x) \in A(X)[t]$, where the ring $A(X)$ is the polynomial ring with variables a_{ij}. Substituting A into $f(x)$ we get a polynomial

$$f(A) \in A(X).$$

Note that if we pick an a matrix A whose entries lie in K, then get a polynomial $g(t) \in K[t]$.

We want to prove that the polynomial $f(A)$ is identically zero. We may asssume that K is algebraically closed. As X is irreducible, it suffices to prove that there is a non-empty open subset $U \subset X$ where this polynomial is the zero function. Now if A is a diagonal matrix, then

$$g(t) = (t - \lambda_1)(t - \lambda_2) \dots (t - \lambda_n),$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the entries on the diagonal. In this case

$$g(A) = (A - \lambda_1 I)(A - \lambda_2 I) \dots (A - \lambda_n I).$$

Now to show that this matrix is the zero matrix, it suffices to show that it gives the zero linear transformation. Almost by definition, if $e_1, e_2, \dots, e_n$ is the standard basis, then $(A - \lambda_i)e_i = 0$, so that the RHS is zero when applied to the standard basis. Thus $g(A)$ is indeed the zero matrix. More generally, this result remains true for any diagonalisable matrix. On the other hand, the matrix A is certainly diagonalisable if the polynomial $g(t)$ has no repeated roots. Indeed in this case, A has n distinct eigenvalues and by a well-known argument, it follows that there is a basis of eigenvectors. So it suffices to prove that the set U consisting of those matrices A such that $g(t)$ has no repeated roots is a non-empty open subset.

The fact that U is non-empty is trivial. Indeed, any diagonal matrix with distinct entries on the diagonal lies in U. To finish off then, it suffices to show that if we have a polynomial $h(t) = h_0 + h_1 t + \dots + h_n t \in K[t]$ of degree n, then the set of points $(h_0, h_1, \dots, h_n)$ where h has a repeated root is a Zariski closed subset of $\mathbb{A}^{n+1}$.

First note that this is equal to the set of points where h and its derivative k have a common zero. Now use the theory of resultants, which yields a (complicated) polynomial involving the coefficients of h and k, to determine the common zeroes.